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ON THE WEYL TENSOR OF A SELF-DUAL COMPLEX 4-MANIFOLD

FLORIN ALEXANDRU BELGUN

Abstract. We study complex 4-manifolds with holomorphic self-dual conformal structures, and we obtain an interpretation of the Weyl tensor of such a manifold as the projective curvature of a field of cones on the ambitwistor space. In particular, its vanishing is implied by the existence of some compact, simply-connected, null-geodesics. We also show that the projective structure of the β-surfaces of a self-dual manifold is flat. All these results are illustrated in detail in the case of the complexification of $\mathbb{CP}^2$.

1. Introduction

Twistor theory, created by Penrose [16], establishes a close relationship between conformal Riemannian geometry in dimension 4, and (almost) complex geometry in dimension 3. In particular, to a Riemannian manifold M for which the part W^- of the Weyl tensor vanishes identically (*self-dual*), one associates its *twistor space* Z, a complex 3-manifold containing rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (called *twistor lines*), and admitting a real structure with no fixed points [1], [6], [3].

The space of such curves is a complex 4-manifold $\mathbf{M}$ [10] with a holomorphic conformal structure and is, therefore, a conformal complexification of M [1], [6], [3].

As the conformal geometry of M is encoded by the complex geometry of Z, we ask ourselves what holomorphic object on Z corresponds to W^+, the Weyl tensor of the self-dual manifold M. It seems that this question, although natural, has not been considered in the literature, and maybe a reason for that is that the answer appears to be a highly non-linear object.

This object is more easily understood in the framework of complex-Riemannian geometry (see Section 2): For a self-dual (complex) 4-manifold $\mathbf{M}$, its (local) twistor space is then defined as the 3-manifold of β-surfaces (some totally geodesic isotropic surfaces; see Section 2). Following LeBrun [14], we further introduce the (locally-defined) space B of complex null-geodesics of $\mathbf{M}$ (*ambitwistor space*).

The ambitwistor space B and (in the self-dual case) the twistor space Z completely describe the conformal structure of $\mathbf{M}$. In particular, a null-geodesic γ in $\mathbf{M}$ corresponds to the set of twistor lines in Z tangent to a 2-plane [13]. The union of these curves, called the *integral α-cone* of γ (see Section 3), is lifted to a (linearized) *α-cone* in $T_\gamma B$. Our first result (Theorem 1) is that the Weyl tensor of $\mathbf{M}$ is equivalent to the projective curvature (see Section 4) of the field of α-cones

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on B. In particular, if such a cone is *flat*, then W^+ vanishes on certain isotropic planes in $\mathbf{M}$.

We use Theorem 1 to investigate global properties of a self-dual manifold $\mathbf{M}$: If the integral α-cone of γ is part of a smooth surface in Z, then the linearized α-cone is flat (Theorems 2, 2′). In particular, the space $\mathbf{M}_0$ of rational curves of Z with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ is compact iff $Z \simeq \mathbb{CP}^3$. On the other hand, it is known, from a theorem of Campana [4], that, for a compact twistor space Z, $\mathbf{M}_0$ can be compactified within the space of analytic cycles iff Z is Moishezon. It appears then that the conformal structure does not extend smoothly to the compactification.

A good illustration of what happens in the non-flat (self-dual) case is the Kähler-Einstein manifold $\mathbb{CP}^2$ whose twistor space is known to be the manifold of flags in $\mathbb{C}^3$ [1]; see Section 7.

A first application of Theorem 2 is that a *civilized* (a topological assumption on the conformal manifold, permitting the global construction of its (ambi-)twistor space—see [12], [15], and Section 2) self-dual manifold containing a compact null-geodesic is conformally flat, and the null-geodesic is simply-connected (Theorem 3). If we assume, in addition, that the compact null-geodesic is simply-connected, then the above result can be deduced, using Theorem 2′, for any self-dual 4-manifold, and also (using the *LeBrun correspondence* [12]) for the case of a conformal complex 3-manifold [2]; In fact, we have recently proven [2] that the existence of a compact, simply-connected, null-geodesic on an n-dimensional conformal complex manifold implies its conformal flatness, for any $n \geq 3$ (different methods are used for $n > 3$).

Another application of Theorem 2 is that the family of *twistor lines* on a *twistor space* never induces a projective structure on it, unless the twistor space is an open set in $\mathbb{CP}^3$ (Corollary 1).

The isotropic, totally geodesic surfaces (called β-surfaces) in a self-dual manifold $\mathbf{M}$ have a projective structure, given by the null-geodesics of $\mathbf{M}$ contained in it (Section 6). We show that it is *flat* (i.e. locally equivalent to $\mathbb{CP}^2$) (Corollary 2), and we obtain a classification of the compact β-surfaces of a self-dual 4-manifold (Theorem 4).

The paper is organized as follows. In Section 2 we recall the classical results of the twistor theory (especially for complex 4-manifolds), in Section 3 we introduce the α-cones on the (ambi-)twistor space, and, in Section 4, we prove the equivalence between the projective curvature of the latter and the Weyl tensor W^+ of $\mathbf{M}$. Section 5 is devoted to the proof of some results of the type "compactness implies conformal (projective) flatness": Theorems 2, 2′ and 3, mentioned above. We study the projective structure of β-surfaces in Section 6, and we illustrate the above results on the special case of the self-dual manifold $\mathbb{CP}^2$ in Section 7.

2. Preliminaries

The content of this paper makes use of *complex-Riemannian geometry*, which is obtained by analogy from Riemannian geometry by replacing the field $\mathbb{R}$ by $\mathbb{C}$ (e.g. a *complex metric* is a non-degenerate symmetric complex-bilinear form on the tangent space), and all classical results hold, naturally with the exception of those making use of partitions of unity. We will often omit the prefix *complex-* when referring to geometric objects, and we will always consider them, unless otherwise stated, in the framework of complex-Riemannian geometry.

2.1. **Conformal complex 4-manifolds.** Let $\mathbf{M}$ be a 4-dimensional complex manifold. A *conformal structure* is defined, as in the real case [5], by an everywhere non-degenerate section c of the complex bundle $S^2(T^*\mathbf{M}) \otimes L^2$, where L is a given line bundle of *scalars of weight 1*, and $L^4 \simeq \kappa^{-1}$, the anti-canonical bundle of $\mathbf{M}$. (While on an oriented real manifold such a line bundle always exists, being topologically trivial, in the complex case the existence of L^2, a square root of the anti-canonical bundle, is submitted to some topological restrictions.) From now on, only holomorphic conformal structures will be considered; thus L is a holomorphic bundle and c a holomorphic section of $S^2(T^*\mathbf{M}) \otimes L^2$. (In fact, all we need to define the conformal structure c on the 4-manifold $\mathbf{M}$ is just the holomorphic bundle L^2; in odd dimensions the situation is different; see [2].)

As in the real case, c is locally represented by symmetric bilinear forms on $T\mathbf{M}$, or local sections in L^2, but global representative metrics do not exist, in general.

For each point $x \in \mathbf{M}$, there is an isotropy cone C_x in the tangent space $T_x\mathbf{M}$, which uniquely determines the conformal structure c. In the associated projective space, $\mathbb{P}(T_x\mathbf{M}) \simeq \mathbb{CP}^3$, the cone C_x projects onto the non-degenerate quadratic surface $\mathbb{P}(C_x)$, which is actually a ruled surface isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. We thus get 2 families of complex projective lines contained in $\mathbb{P}(C)$, that is, 2 families of isotropic 2-planes in $C \subset T\mathbf{M}$, respectively called α-planes and β-planes. This choice corresponds to the choice of an *orientation* of $\mathbf{M}$. On a real 4-manifold an orientation is chosen by picking a class of "positive" volume forms (which is not possible in this complex framework) or by choosing one of the two possible Hodge operators compatible with the conformal structure $* : \Lambda^2\mathbf{M} \to \Lambda^2\mathbf{M}$ (which can also be done in our complex case, [17]). As $*$ is a symmetric involution, $\Lambda^2\mathbf{M}$ decomposes into $\Lambda^+\mathbf{M} \oplus \Lambda^-\mathbf{M}$, consisting in ±1–eigenvectors of $*$, respectively called *self-dual* and *anti-self-dual* 2-forms; the isotropic vectors in $\Lambda^+\mathbf{M}$ and $\Lambda^-\mathbf{M}$ are then exactly the *decomposable* elements $u \wedge v \in \Lambda^{\pm}\mathbf{M}$, with $u, v \in \mathbf{M}$.

Definition 1. An *α-plane* F^α (resp. a *β-plane* F^β) in $T\mathbf{M}$ is a 2-plane such that $\Lambda^2 F^\alpha$ (resp. $\Lambda^2 F^\beta$) is a self-dual (resp. anti-self-dual) isotropic line in $\Lambda^2\mathbf{M}$.

Remark. The α- and β-planes can be interpreted in terms of spinors. The structure group of the tangent bundle $T\mathbf{M}$ is restricted to the conformal orthogonal complex group, $CO(4,\mathbb{C}) := (O(4,\mathbb{C}) \times \mathbb{C}^*)/\{\pm\mathbf{1}\}$, where $O(4,\mathbb{C}) := \{A \in GL(4,\mathbb{C}) | A\, {}^tA = \mathbf{1}\}$, by the given conformal structure of $\mathbf{M}$. The choice of an orientation is the further restriction of this group to the connected component of the identity, $CO_0(4,\mathbb{C}) := SO(4,\mathbb{C}) \times \mathbb{C}^*$, where $SO(4,\mathbb{C}) := O(4,\mathbb{C}) \cap SL(4,\mathbb{C})$. Consider a local metric g in the conformal class c. We have then locally defined *Spin* structures, and associated *Spin* bundles V_+, V_-, as in the real case [1],[18]. They are rank 2 complex vector bundles, and for each local section of L^2 (i.e. a metric in c), each of them is equipped with a (complex) symplectic structure $\omega_+ \in \Lambda^2 V_+, \omega_- \in \Lambda^2 V_-$, respectively. Then we locally have $T\mathbf{M} \simeq V_+ \otimes V_-$, and $g = \omega_+ \otimes \omega_-$, for the fixed metric $g \in c$. α- (resp. β-) planes are then nothing but the isotropic 2-planes obtained by fixing the first (resp. the second) factor in the above tensor product:

Proposition 1 ([17]). *An α-plane, resp. β-plane $F \subset T_x\mathbf{M}$ is a complex plane $\psi_+ \otimes V_-$, resp $V_+ \otimes \psi_-$, where $\psi_+ \in V_+ \setminus \{0\}$, resp. $\psi_- \in V_- \setminus \{0\}$.*

The α-planes in $T_x\mathbf{M}$ are thus indexed by $\mathbb{P}(V_+)_x$, and β-planes by $\mathbb{P}(V_-)_x$, and these projective bundles are globally well-defined on $\mathbf{M}$ [1].

Remark. It is obvious that a change of orientation interchanges the α- and β-planes; the same is true for self-duality and anti-self-duality, to be defined below.

For a local metric g in c, we denote by R^g its Riemannian curvature, and by W the Weyl tensor, i.e. the trace-free component of R^g, which is known to be independent of the chosen metric within the conformal class [5]. It splits into two components W^+, W^-, and the easiest way to see that is the spinorial decomposition of the space of the curvature tensors $\mathcal{R} \subset \Lambda^2 \otimes \Lambda^2$ ([1], [18], [19]), obtained from the relation $T\mathbf{M} = V_+ \otimes V_-$ and some of the Clebsch-Gordan identities [18]. We have

$$\mathcal{R} = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{W}^+ \oplus \mathcal{W}^-,$$

where $\mathcal{S}$ is the complex line of scalar curvature tensors, ("diagonally") included in $\Lambda^2 V_+ \oplus \Lambda^2 V_- \simeq \mathbb{C} \oplus \mathbb{C}$, $\mathcal{B} = S^2 V_+ \otimes S^2 V_-$ is the space of trace-free Ricci tensors, and $\mathcal{W}^+ = S^4 V_+$, $\mathcal{W}^- = S^4 V_-$ are the spaces of self-dual, resp. anti-self-dual Weyl tensors (where $S^p V_\pm$ denotes the p-symmetric power of $V_\pm$).

The curvature R^g restricted to any α-plane F yields a weighted bilinear symmetric form R^F on $\Lambda^2 F$, i.e. a section in $L^2 \otimes (\Lambda^2 F \otimes \Lambda^2 F)^*$:

$$(g, X \wedge Y) \overset{R^F}{\longmapsto} g(R^g(X,Y)X,Y).$$

Proposition 2. *The (weighted) bilinear form R^F depends only on the self-dual Weyl tensor, and this one is completely determined by the (weighted) values of R^F for all α-planes F.*

We have the same result for β-planes.

Proof. Let $F = \psi_+ \otimes V_-$ be an α-plane, let $X = \psi_+ \otimes \varphi_1, Y = \psi_+ \otimes \varphi_2 \in F$, and suppose, for simplicity, that $\omega_-(\varphi_1, \varphi_2) = 1$, so $X \wedge Y \in \Lambda^2 F$ is identified with the element $\psi_+ \otimes \psi_+ \in S^2 V_+$. Then it is easy to see that R^F, evaluated on $X \wedge Y$, is nothing but the evaluation of $R \in S^2(\Lambda^2 \mathbf{M}) \supset \mathcal{R}$ on $(X \wedge Y) \otimes (X \wedge Y) \simeq \psi_+ \otimes \psi_+ \otimes \psi_+ \otimes \psi_+ \in S^4 V_+$, which depends only on the positive (or self-dual) part of the Weyl tensor. To prove the second assertion, we remark that W^+, being a quadrilinear symmetric form on V_+, can be identified with a polynomial of degree 4 on V_+, which is determined by its values. □

Definition 2. A conformal structure c on a 4-manifold $\mathbf{M}$ is called *self-dual* (resp. *anti-self-dual*) iff $W^- = 0$ (resp. $W^+ = 0$).

Remark. In general, geodesics on a conformal manifold depend on the chosen metric, with the exception of the isotropic ones (or *null-geodesics*). Therefore the existence of totally geodesic surfaces tangent to α- (resp. β-) planes is a property of the conformal structure alone.

2.2. **Twistor spaces.**

Definition 3. An *α-surface* (resp. *β-surface*) $\alpha \subset \mathbf{M}$ is a maximal, totally geodesic surface in $\mathbf{M}$ whose tangent space at any point is an α-plane (resp. β-plane).

On the other hand, any totally geodesic, isotropic surface in $\mathbf{M}$ is included in an α- or in a β-surface.

Definition 4 ([16], [17])**.** If, at any point $x \in \mathbf{M}$, and for any α- (resp. β-) plane $F \subset T_x\mathbf{M}$, there is an α- (resp. β-) surface tangent to F at x, we say that the family of α- (resp. β-) planes is *integrable.*

Theorem ([1], [17]). *The family of α- (resp. β-) planes of a conformal 4-manifold* $(\mathbf{M}, c)$ *is integrable if and only if the conformal structure c is anti-self-dual (resp. self-dual).*

The integrability of α-planes is equivalent to the integrability (in the sense of Frobenius) of a distribution H^α of 2-planes on the total space of the projective bundle $\mathbb{P}(V_+)$. Namely, let g be a local metric in the conformal class c, and let ∇ be its Levi-Civita connection. ∇ induces a connection in the bundle $\mathbb{P}(V_+)$, thus a horizontal distribution H, isomorphic to $T\mathbf{M}$ via the bundle projection. Let H^α be the 2-dimensional subspace of H_F—where $F \in \mathbb{P}(V_+)$ is an α-plane in $T_x\mathbf{M}$—which projects onto $F \subset T_x\mathbf{M}$. It can easily be shown (as in [17], see also [1]) that the *tautological* 2-plane distribution H^α is independent of the metric g. Then α-surfaces are canonically lifted as integrable manifolds of the distribution H^α. For a geodesically convex open set of $\mathbf{M}$, one can prove (see [15]) that the space of these integrable leaves is a complex 3-manifold. (This point of view is closely related to that of [1], about the integrability of the canonical almost complex structure of the real twistor space.)

The same remark can be made about β-surfaces.

Remark. The existence, for any point $x \in \mathbf{M}$, of an α-surface containing x does not imply, in general, the integrability of the whole family of α-planes: in the conformal self-dual (but not anti-self-dual) manifold $\mathbf{M} = \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \smallsetminus \mathcal{F}$ (the complexification of $\mathbb{CP}^2$, [1]), the surfaces $(\{x\} \times (\mathbb{CP}^2)^*) \cap \mathbf{M}$ and $(\mathbb{CP}^2 \times \{y\}) \cap \mathbf{M}$ are all α-surfaces, see Section 7.

Remark. In the real framework, the twistor space of a real Riemannian 4-manifold $M^\mathbb{R}$ is the total space $Z^\mathbb{R}$ of the S^2-bundle of almost-complex structures on $TM^\mathbb{R}$, compatible with the conformal structure and the (opposite) orientation; it admits a natural almost-complex structure $\mathcal{J}$, equal, at the point $J \in Z^\mathbb{R}$, to the complex structure of the fibers on the *vertical* space $T_J^\vee Z^\mathbb{R}$, and to J itself on the horizontal space (induced by the Levi-Civita connection). Such a complex structure J is equivalent to an isotropic complex 2-plane in $TM \otimes \mathbb{C}$, thus to an α- or β-surface (depending on the conventions), which then becomes the space of vectors of type $(1,0)$ for J; as the integrability of the almost-complex structure $\mathcal{J}$ can be expressed as the Frobenius condition applied to $T^{(1,0)}Z^\mathbb{R}$, it is equivalent to the integrability of the family of α-, resp. β-planes.

The *Penrose construction* associates to an (anti-)self-dual manifold $\mathbf{M}$ the space Z of α- (resp. β-)surfaces of $\mathbf{M}$; we have seen above that Z admits complex-analytic maps, but it may be non-Hausdorff. This is why we need to introduce the following condition; see also [15]:

Definition 5. An (anti-)self-dual manifold $\mathbf{M}$ is called *civilized* iff the space Z^α (resp. Z^β) of integral leaves of the distribution H^α (resp. H^β) in $\mathbb{P}(V_+)$ (resp. $\mathbb{P}(V_-)$) is a complex 3-manifold, and the projection $p^+ : \mathbb{P}(V_+) \to Z^\alpha$ (resp. $p^- : \mathbb{P}(V_-) \to Z^\beta$) is a submersion.

In this case, the manifold Z^α (resp. Z^β)—which is the space of α-surfaces (resp. β-surface) of $\mathbf{M}$ — is called the α- (resp. the β-)*twistor space* of $\mathbf{M}$.

From now on, we suppose that $(\mathbf{M}, c)$ is a self-dual complex analytic 4-manifold. As any point $x \in \mathbf{M}$ has a geodesically convex neighborhood U [21] (which is, therefore, civilized [15]), we can construct Z^U, the β-twistor space (for short, twistor space) of U. As most results of this paper are infinitesimal, we will usually suppose (with no loss of generality) that $\mathbf{M}$ is civilized (for example, by replacing $\mathbf{M}$ by U).

We recall now the correspondence between differential geometric objects on $\mathbf{M}$ and complex analytic objects on its twistor space Z ([1], [17]; see also [12], [13], [18]).

β-surfaces $\beta \subset \mathbf{M}$ correspond to points $\bar{\beta} \in Z$, by definition, and the set of β-surfaces passing through a point $x \in \mathbf{M}$ is a complex projective line Z_x, with normal bundle isomorphic (non-canonically) to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (where $\mathcal{O}(1)$ is the dual of the *tautological* bundle $\mathcal{O}(-1)$ on $\mathbb{CP}^1$) ([1], [17]; see also [3]). Such a curve will be called a *twistor line*.

In fact, the family of twistor lines in Z permits us to recover $\mathbf{M}$ and its conformal structure, at least locally, by the *reverse Penrose construction*: The normal bundle N_x of a line Z_x in Z has the property $H^1(N_x, \mathcal{O}) = 0$; thus, by a theorem of Kodaira [10], the space $\mathbf{M}_0$ of projective lines in Z having the above normal bundle is a smooth complex manifold whose tangent space at a point $x \simeq Z_x \subset Z$ is canonically isomorphic to the space of global sections of the normal bundle N_x of Z_x (thus $\mathbf{M}_0$ has dimension 4). The conformal structure of $\mathbf{M}_0$ is described by its isotropy cone, which corresponds to the sections of N_x having at least one zero (as such a section decomposes as 2 sections of $\mathcal{O}(1)$, the vanishing condition means that they both vanish at the same point, which is a quadratic condition on the sections of N_x). We thus get a conformal diffeomorphism from $\mathbf{M}$ to an open set of $\mathbf{M}_0$.

2.3. **Ambitwistor spaces.** We remark that $\mathbb{P}(V_-)$ is an open set of the projective tangent bundle of Z, as Z is the space of leaves of $\mathbb{P}(V^-)$, but it is important to note that, in general, the reverse inclusion is not true (i.e. not every direction in Z is tangent to a line corresponding to a point in $\mathbf{M}$, or, equivalently, β-surfaces are not compact $\mathbb{CP}^2$'s, in general, see Section 5).

For example, if $\mathbf{M} = \mathbb{CP}^2 \times \mathbb{CP}^* \setminus \mathcal{F}$ (with the notation in Section 7), $\mathbb{P}(V_-)$ is an open subset in the $\mathbb{CP}^2$-bundle $\mathbb{P}(TZ) \to Z$, consisting of the set of directions transverse to the *contact structure* of Z (see Subsection 7.4). $\mathbb{P}(V_-)$ is thus, in this case, a rank 2 affine bundle over Z.

Another canonical $\mathbb{CP}^2$-bundle on Z, that is, $\mathbb{P}(T^*Z) \to Z$, leads to the *ambitwistor space* B, which is by definition the space of null-geodesics of $\mathbf{M}$ [13]. It is an open set of the projective cotangent bundle of Z (or, equivalently, the Grassmannian of 2-planes in TZ) [13]. More precisely, a plane $F \subset T_{\bar{\beta}}Z$ corresponds to a null-geodesic $\gamma \subset \mathbf{M}$ (contained in β) if it is tangent to at least one projective line Z_x, corresponding to a point $x \in \mathbf{M}$.

To see that, let x be a point in $\mathbf{M}$, β a β-surface passing through x, i.e. $\bar{\beta} \in Z$ and Z_x contains $\bar{\beta}$; let $F \subset T_{\bar{\beta}}Z$ be a plane tangent to Z_x. As small deformations of Z_x still correspond to points of $\mathbf{M}$, we consider the twistor lines which are tangent to F. They correspond to a (path-connected) set of points on a curve $\gamma \subset \beta$, which will turn out to be a null-geodesic. Indeed, all we have to prove is $\ddot{\gamma} = 0 \pmod{\dot{\gamma}}$, and $\dot{\gamma}_x$ corresponds to a section η of N_x, vanishing at $\bar{\beta} \in Z_x$; as $N_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$, η is determined by its derivative at $\bar{\beta}$, which is a linear map $T_{\bar{\beta}} \to F/T_{\bar{\beta}}$ (the infinitesimal deformation of the direction of Z_x within F). As the points of γ correspond to lines tangent to F, we have that $\ddot{\gamma}_x$ corresponds to a section of N_x collinear to η; thus γ satisfies the equation of a (non-parameterized) geodesic. See [12], [15], and Section 4 for details.

Example. The space of null-geodesics of $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ is the total space of a $\mathbb{C} \times \mathbb{CP}^1$-bundle over $Z = \mathcal{F}$, the flag manifold (see Section 7); a 2-plane $F \subset T_{(L,l)}\mathcal{F}$ which corresponds to a null-geodesic in $\mathbf{M}$ is identified either with a projective diffeomorphism $\varphi : \mathbb{P}(l) \to \mathbb{P}(L^o)$ (Subsection 7.4, case 3), or with a point $A \subset l$, $A \neq L$, resp. a plane a containing L, and different from l (Subsection 7.4, cases **2** and **2′**).

3. The structure of the ambitwistor space and the field of α-cones

Conventions. Except for some results in Section 5, we will consider $\mathbf{M}$ to be a self-dual civilized 4-manifold, i.e. the (twistor) space Z of β-surfaces of $\mathbf{M}$ is a Hausdorff smooth complex 3-manifold, and the projection $\mathbb{P}(V_-) \to Z$ is a submersion (e.g. $\mathbf{M}$ is geodesically convex); see [15].

We will frequently identify, following the deformation theory of Kodaira (see [10]), the vectors in $T_x\mathbf{M}$ with sections in the normal bundle $N(Z_x)$ of the projective line Z_x in Z.

We also consider the space of null-geodesics B, as an open subset of $\mathbb{P}(T^*Z)$.

For a null-geodesic γ, resp. a β-surface $\beta \subset \mathbf{M}$, we denote by $\bar{\gamma}$, resp. $\bar{\beta}$, the corresponding point in B, resp. Z.

3.1. α- and β-cones on the ambitwistor space.

The vectors on B can be expressed in terms of infinitesimal deformations of geodesics of $\mathbf{M}$ (Jacobi fields). More precisely,

$$T_{\bar{\gamma}}B \simeq \mathcal{J}_\gamma^{\perp}/\mathcal{J}_\gamma^{\gamma},$$

where, for a null-geodesic γ, $\mathcal{J}_\gamma^{\perp}$ is the space of Jacobi fields J such that $\nabla_{\dot{\gamma}}J \perp \dot{\gamma}$, and $\mathcal{J}_\gamma^{\gamma}$ is its subspace of Jacobi fields "along" γ, i.e. $J \in \mathbb{C}\dot{\gamma}$ at any point of the geodesic.

Remark. A class in $\mathcal{J}_\gamma^{\perp}/\mathcal{J}_\gamma^{\gamma}$ is represented by Jacobi fields yielding the same local section of the normal bundle $N(\gamma)$ of γ in $\mathbf{M}$. This is equivalent to the following obvious fact:

Lemma 1. *The kernel of the natural application $\mathcal{J}_\gamma^{\perp} \to N(\gamma)$ is $\mathcal{J}_\gamma^{\gamma}$.*

As a consequence, Jacobi fields on γ induce particular local sections in $N(\gamma)$, which turn out to be (conformally invariant) solutions of a differential operator of order 2 on $N(\gamma)$; see [2].

The conformal geometry of $\mathbf{M}$ induces a particular structure on B: we describe it in order to obtain an expression of W^+ in terms of the geometry of the (ambitwistor) space.

We have a canonical hyperplane $V_{\bar{\gamma}}$ in $T_{\bar{\gamma}}B$, defined by

$$V_{\bar{\gamma}} := \mathcal{J}_\gamma^{\perp\perp}/\mathcal{J}_\gamma^{\gamma},$$

where $\mathcal{J}_\gamma^{\perp\perp}$ is the set of Jacobi fields J everywhere orthogonal to $\dot{\gamma}$ (i.e. $\nabla_{\dot{\gamma}}J \perp \dot{\gamma}$ *and* $J \perp \dot{\gamma}$).

Now we define two fields of cones in TB, both contained in $V_{\bar{\gamma}}$:

Definition 6. Let γ be a null-geodesic in $\mathbf{M}$, and, for each point $x \in \gamma$, let F_x^β be the β-plane containing $\dot{\gamma}_x$. The (*infinitesimal*) β-*cone* $V_{\bar{\gamma}}^\beta$ *at* $\bar{\gamma} \in B$ is defined as follows:

$$V_{\bar{\gamma}}^\beta := \mathcal{J}_\gamma^\beta/\mathcal{J}_\gamma^\gamma \subset \mathcal{J}_\gamma^{\perp\perp}/\mathcal{J}_\gamma^\gamma \simeq V_{\bar{\gamma}} \subset T_{\bar{\gamma}}B,$$

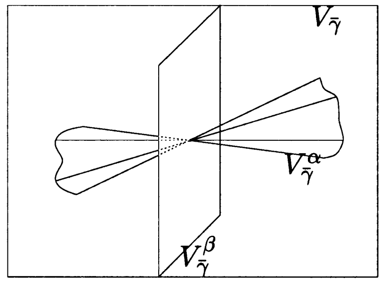

Figure 1.

where $\mathcal{J}^{\beta}_{\gamma}$ is the set of Jacobi fields J on γ satisfying the condition

$$\exists\, x \in \gamma \text{ such that } J_x = 0 \text{ and } (\nabla_{\dot{\gamma}} J)_x \in F^{\beta}_x.$$

Proposition 3. *The β-cone $V^{\beta}_{\bar{\gamma}}$ is* flat, *i.e. it is included in the 2-plane $F^{\beta}_{\bar{\gamma}}$ consisting of Jacobi fields contained in the β-plane defined by $\dot{\gamma}$ at each point of it.*

Proof. We have to prove that $\mathcal{J}^{\beta}_{\gamma}$ is included in $\bar{\mathcal{J}}^{\beta}_{\gamma}$, defined as follows:

$$\bar{\mathcal{J}}^{\beta}_{\gamma} := \{J \text{ a Jacobi field on } \gamma \mid J_x, \dot{J}_x \in F^{\beta}_x,\ \forall x \in \gamma\}.$$

We will prove that $\mathcal{J}^{\beta}_{\gamma} \subset \bar{\mathcal{J}}^{\beta}_{\gamma}$; therefore it will follow that the latter is non-empty, and is a linear space of dimension 2.

We denote by K^0 the parallel displacement, along γ, of a non-zero vector in F^{β}_x, transverse to $\dot{\gamma}$. Then $K^0 \in T\beta|_{\gamma} \smallsetminus T\gamma$, because γ is included in the totally geodesic surface β; thus we can characterize F^{β}_y as the set $\{X \in T_yM \mid X \perp \dot{\gamma}, X \perp K^0\}$, for any $y \in \gamma$. We then observe that

$$\dot{\gamma}.\langle \dot{J}, K^0\rangle = \langle R(\dot{\gamma}, J)\dot{\gamma}, K^0\rangle = \langle R(\dot{\gamma}, K^0)\dot{\gamma}, J\rangle = k\langle K^0, J\rangle,$$

because $R(\dot{\gamma}, K^0)\dot{\gamma}$ is in F^{β}, thus $R(\dot{\gamma}, K^0)\dot{\gamma} = h\dot{\gamma} + kK^0$. So the scalar function $\langle J, K^0\rangle$ satisfies a linear second order equation, and hence it it determined by its initial value and derivative. It follows then that it is identically zero; thus $J \in F^{\beta}$ everywhere, as claimed. □

Another subset in $T_{\bar{\gamma}}B$ is the α-cone $V^{\alpha}_{\bar{\gamma}}$, defined as follows:

Definition 7. Let γ be a null-geodesic in **M**, and, for each point $x \in \gamma$, let F^{α}_x be the α-plane containing $\dot{\gamma}_x$. The (*infinitesimal*) *α-cone $V^{\alpha}_{\bar{\gamma}}$ at $\bar{\gamma} \in B$* is defined as follows:

$$V^{\alpha}_{\bar{\gamma}} := \mathcal{J}^{\alpha}_{\gamma}/\mathcal{J}^{\gamma}_{\gamma} \subset \mathcal{J}^{\perp\perp}_{\gamma}/\mathcal{J}^{\gamma}_{\gamma} \simeq V_{\bar{\gamma}} \subset T_{\bar{\gamma}}B,$$

where $\mathcal{J}^{\alpha}_{\gamma}$ is the set of Jacobi fields J on γ satisfying the condition

$$\exists x \in \gamma \text{ such that } J_x = 0 \text{ and } (\nabla_{\dot{\gamma}} J)_x \in F^{\alpha}_x.$$

It is important to note that, in general, the projective curves $\mathbb{P}(V^{\alpha}_{\bar{\gamma}})$ and $\mathbb{P}(V^{\beta}_{\bar{\gamma}})$ are non-compact, as each of them corresponds to the set of points on γ, which is non-compact, in general. The field of α-cones on B is the object of main interest in this paper. We may already guess that its flatness (i.e. the situation when $V^{\alpha}_{\bar{\gamma}}$ is a subset in a 2-plane) can be related to some vanishing property of the self-dual Weyl tensor of **M**. See Figure 1.

Remark. We have seen that $V^\beta_{\bar\gamma}$ is included in the 2-plane $F^\beta_{\bar\gamma}$, i.e. the condition $J_x = 0, \dot J_x \in F_x$ can be generalized to the linear condition $J, \dot J \in F^\beta$, but there is no canonical way of supplying the "missing" points of γ with some appropriate Jacobi fields in order to "complete" $V^\alpha_{\bar\gamma}$ as in the β-cones case. This would be possible, for example, if $\mathbb{P}(V^\alpha_{\bar\gamma})$ were an open subset in a projective line. But the failure of $V^\alpha_{\bar\gamma}$ to be part of a 2-plane is measured by its *projective curvature*, and we will see in Section 4 that the vanishing of the latter implies the vanishing of W^+ (Theorem 1).

3.2. **Integral α-cones in Z and B.** Now we study the field of α-cones of B in relation with Z and the canonical projection $\pi : B \to Z$. First, we note that there are complex projective lines in B tangent to the directions in $V^\alpha_{\bar\gamma}$:

Definition 8. Let $\bar\gamma \in B$, and let $x \in \gamma$ be a point on the null-geodesic γ; let F^α_x be the α-plane tangent to $T_x\gamma$. The *rational curve* $B^\alpha_{\bar\gamma,x}$ *in* B (*containing* $\bar\gamma$) is by definition the set of null-geodesics passing through x and tangent to F^α_x.

The curves $B^\alpha_{\bar\gamma,x}$, $x \in \gamma$, are projected by π onto the complex lines Z_x through $\bar\beta$ (corresponding to the β-surface β containing γ) tangent to the 2-plane F^γ.

On the other hand, it is easy to see that the complex projective lines $B^\beta_{\bar\gamma,x}$ (defined analogously to $B^\alpha_{\bar\gamma,x}$), which are tangent to (an open set of the directions of) $V^\beta_{\bar\gamma}$, are contained in the fibers of π. In fact, they coincide with some of the projective lines passing through the point $\gamma \in \mathbb{P}(T^*_{\bar\beta}Z) \simeq \mathbb{CP}^2$.

Definition 9. The *integral α-cones* in B, resp. Z, are defined by:

$$B^\alpha_{\bar\gamma} := \bigcup_{x\in\gamma} B^\alpha_{\bar\gamma,x}\ (\beta\text{-cone in } B); \quad Z^\gamma := \bigcup_{x\in\gamma} Z_x\ (\beta\text{-cone in } Z).$$

We intend to prove that $B^\alpha_{\bar\gamma}$ is the *canonical lift* of Z^γ (see Proposition 5). We know that $\pi(B^\alpha_{\bar\gamma}) = Z^\gamma$. We have

Proposition 4. *Except for the vertices $\bar\gamma \in B^\alpha_{\bar\gamma}$ and $\bar\beta \in Z^\gamma$, the two integral cones $B^\alpha_{\bar\gamma}$ and Z^γ are smooth, immersed surfaces of B, resp. Z.*

Proof. The open set B of $\mathbb{P}(T^*Z)$ which is the space of null-geodesics of $\mathbf{M}$ can be viewed as the space of integral curves of the *geodesic distribution* G of lines in $\mathbb{P}(C)$, the total space of the fibre bundle of isotropic directions in $T\mathbf{M}$. G_v is defined as the horizontal lift (for the Levi-Civita connection on $\mathbf{M}$) of v, which is an isotropic line in $T_x\mathbf{M}$. This definition is independent of the chosen metric and connection [15], and, by integrating this distribution (as $\mathbf{M}$ is civilized), we get a holomorphic map $p : \mathbb{P}(C) \to B$, where B is the space of leaves of this foliation. This map can be used to compute the normal bundle of $B^\alpha_{\bar\gamma,x}$, $N(B^\alpha_{\bar\gamma,x})$; see [12], [13], [15].

Indeed, we have lines $C^\alpha_{\gamma,x} \in \mathbb{P}(C)_x$, such that $\dot\gamma_x \in C^\alpha_{\gamma,x}$, which project onto $B^\alpha_{\bar\gamma,x}$; thus we get the following exact sequence of normal bundles:

$$0 \to N(C^\alpha_{\gamma,x}; p^{-1}(B^\alpha_{\bar\gamma,x})) \to N(C^\alpha_{\gamma,x}; \mathbb{P}(C)) \to N(B^\alpha_{\bar\gamma,x}; B) \to 0,$$

where we have written the ambient spaces of the normal bundles on the second position. The central bundle is trivial ($C^\alpha_{\gamma,x}$ is trivially embedded in $\mathbb{P}(C)_x \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, which is trivially embedded in $\mathbb{P}(C)$ as a fibre), and it is easy to check that the left-hand bundle is isomorphic to the tautological bundle over $\mathbb{CP}^1$, $\mathcal{O}(-1)$. This proves that $N(B^\alpha_{\bar\gamma,x}; B) \simeq \mathcal{O}(0) \oplus \mathcal{O}(0) \oplus \mathcal{O}(1)$; in particular, the conditions in the completeness theorem of Kodaira [10] are satisfied. Thus the lines in the integral α-cone $B^\alpha_{\bar\gamma}$ form an analytic subfamily of the family $\{B^\alpha_{\bar\gamma,x}\}_{\bar\gamma\in B, x\in\gamma\subset M}$ that correspond to the sections of the normal bundle of $B^\alpha_{\bar\gamma,x}$ vanishing at $\bar\gamma \in B$, or, equivalently, to the points x of $\gamma \subset \mathbf{M}$.

But, in order to prove the smoothness of $B^\alpha_{\bar\gamma} \smallsetminus \{\bar\gamma\}$, we first remark that the surface $C^\alpha_\gamma \subset \mathbb{P}(C)$, defined as follows, is smooth:

$$C^\alpha_\gamma := \{v \in \mathbb{P}(C)_x | x \in \gamma,\ v \subset F^\alpha_\gamma\},$$

where F^α_γ is the α-plane containing $\dot\gamma$. C^α_γ is smooth, and $p(C^\alpha_\gamma) = B^\alpha_{\bar\gamma}$. We note now that C^α_γ is everywhere, except at the points of $p^{-1}(\bar\gamma)$, transverse to the fibers of the submersion $p : \mathbb{P}(C) \to B$. We may conclude that $B^\alpha_{\bar\gamma} \smallsetminus \{\gamma\}$ is a smooth analytic submanifold of B (not closed).

We can use similar methods to prove that $Z^\gamma \smallsetminus \{\bar\beta\}$ is an immersed submanifold of Z (by using the projection $\pi : B \to Z$). □

There is another argument for this latter claim, which gives the tangent space to Z^γ at any point.

We see Z^γ as the *trajectory* of a 1-parameter deformation of Z_x: we fix $\bar\beta$ and we "turn" Z_x around $\bar\beta$ by keeping it tangent to F^γ. The trajectory of this deformation is smooth in $\zeta \in Z^\gamma \smallsetminus \beta$ iff any non-identically-zero section ν of the normal bundle $N(Z_x)$ corresponding to this 1-parameter deformation does not vanish at ζ. In particular, the tangent space $T_\zeta Z^\gamma$ is spanned by $T_\zeta Z_x$ and $\nu(\zeta)$.

But the sections ν generating this deformation are the sections of $N(Z_x)$ vanishing at $\bar\beta$, and they vanish at only one point (and even there, only to order 0) unless they are identically zero, because $N(Z_x) \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$.

Remark. The values of these sections at the points of Z_x other than $\bar\beta$, plus their derivatives at $\bar\beta$ (well-defined as they all vanish at $\bar\beta$), define a 1-dimensional subbundle of $N(Z_x)$ which is isomorphic to $\mathcal{O}(1)$. In fact, we have a 1–1 correspondence between the subbundles of $N(Z_x)$ isomorphic to $\mathcal{O}(1)$ and the 2-planes in $T_{\bar\beta}Z$. Then, the space of holomorphic sections of such a bundle is a linear space of dimension 2, consisting of a family of sections of $N(Z_x)$ vanishing on *different* points of Z_x. Thus we get a 2-plane F^α of isotropic vectors in $T_x\mathbf{M}$, which is easily seen to be an α-plane, as the β-plane $F^\beta_x = T_x\beta$ consists of the set of all sections of $N(Z_x)$ vanishing at $\bar\beta$ (we have $F^\alpha_x \cap T_x\beta = T_x\gamma$). The tangent space to Z^γ at a point $\zeta \in Z_x$ is spanned by the subbundle of $N(Z_x)$ (isomorphic to $\mathcal{O}(1)$—see above) defined by the isotropic vectors $v \in F^\alpha_x$. If γ^ζ is the null-geodesic generated by v^ζ, we conclude that $T_\zeta Z^\gamma$ is the 2-plane determined by γ^ζ, and that $\zeta = \pi(\overline{\gamma^\zeta})$. See Figure 2.

Example. If $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \smallsetminus \mathcal{F}$, then the integral α-cone Z^γ in Z, for $\gamma \equiv F^\gamma = F^\varphi \subset T_{(L,l)}Z$ (where $\varphi : \mathbb{P}(l) \to \mathbb{P}(L^o)$ is a projective diffeomorphism), is the (smooth away from the vertex (L,l)) surface $\{(S, \varphi(s)) | S \neq L, s \neq l, \varphi(s \cap l) = S\}$. Its compactification (by adding the *special cycle* $\bar Z_{(L,l)}$) is singular (Subsection 7.4).

As any smooth surface in Z has a canonical lift in $B = \mathbb{P}(T^*Z)$, we get

Proposition 5. *The integral α-cone $B^\alpha_{\bar\gamma}$ is the canonical lift of the integral α-cone Z^γ on Z. See Figure* 3.

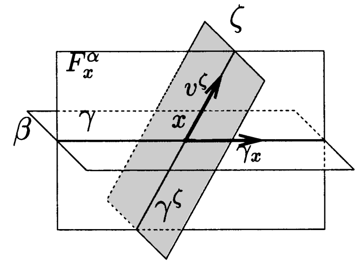

FIGURE 2.

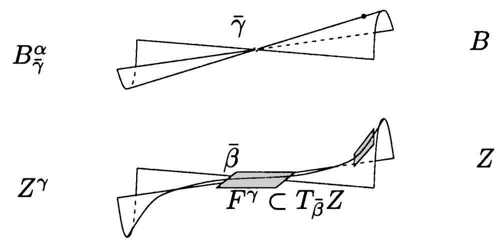

FIGURE 3.

Remark. Basically, this lift can only be defined for $Z^\gamma \smallsetminus \{\bar{\beta}\}$, but in this special case it can be extended by continuity to $\bar{\beta}$. Of course, the smoothness of the lifted surface can only be deduced away from the vertex $\bar{\gamma}$ (from the smoothness of $Z^\gamma \smallsetminus \{\bar{\beta}\}$).

4. THE PROJECTIVE CURVATURE OF THE α-CONE V_γ^α AND THE SELF-DUAL WEYL TENSOR W^+ ON $\mathbf{M}$

As noted in Section 3, we intend to find a relation between the "curvature" of the α-cone $V_{\bar{\gamma}}^\alpha$ (its non-flatness) and the Weyl tensor W^+ of $(\mathbf{M}, c)$. We begin by defining the *projective curvature* of V_γ^α: A projective structure on a manifold X is an equivalence class of linear connections yielding the same geodesics. In such a space, we can define the *projective curvature* of a curve S at a point σ as the linear application $k : T_\sigma S \otimes T_\sigma S \to N(S)_\sigma = T_\sigma X / T_\sigma S$, with $k(Y) := \nabla_Y Y$(modulo $T_\sigma S$), for ∇ any connection in the projective structure of X. In particular, we take for X the projective space $\mathbb{P}(T_{\bar{\gamma}} B)$, with its canonical projective structure, and for S we take $\mathbb{P}(V_{\bar{\gamma}}^\alpha)$, the projectivized α-cone in $\bar{\gamma}$.

Definition 10. The *projective curvature of the α-cone $V_{\bar{\gamma}}^\alpha$ at the generating line $\sigma \subset V_{\bar{\gamma}}^\alpha$* is the projective curvature of $S := \mathbb{P}(V_{\bar{\gamma}}^\alpha)$ in σ, and is identified with a linear application

$$K_{\gamma,x}^\alpha : T_\sigma S \otimes T_\sigma S \to N(S)_\sigma,$$

where σ is the tangent direction to $B_{\bar{\gamma},x}^\alpha$ in $\bar{\gamma}$.

In order to compute the projective curvature of $V_{\bar{\gamma}}^\alpha$, we first establish some canonical isomorphisms between the spaces appearing in the above definition and some linear subspaces of $T_x\mathbf{M}$. We fix the geodesic γ, the point $x \in \gamma$ (therefore also $\sigma = T_{\bar\gamma}B^{\alpha}_{\bar\gamma,x} \in \mathbb{P}(T_{\bar\gamma}B)$), and, thus, the α-plane $F^{\alpha}_x \subset T_x\mathbf{M}$ containing $\dot\gamma_x$, as well as $\dot\gamma_x^{\perp} \subset T_x\mathbf{M}$, the space orthogonal to $\dot\gamma_x$.

For simplicity, in the following lemmas we will omit some indices referring to these fixed objects.

Lemma 2. *There is a canonical isomorphism τ between the tangent space $T_\sigma S$ to the projective cone $S = \mathbb{P}(V^{\alpha}_{\bar\gamma})$ and the tangent space $T_x\gamma$ to the geodesic γ at the point x corresponding to the direction $\sigma \in \mathbb{P}(T_{\bar\gamma}B)$.*

Proof. Let $Y \in T_x\gamma$. We define $\tau^{-1}(Y)$ as follows. Recall that $T_\sigma S \simeq \mathrm{Hom}(\sigma, E/\sigma)$, where $E\ (= E_x) := T_\sigma V^{\alpha}_{\bar\gamma}$ (the tangent space at a point to a cone is the same for all points on the line containing the point). We know that σ corresponds to $\mathcal{J}^{\alpha}_{\gamma,x}$, the space of Jacobi fields on γ vanishing at x and such that $\dot J_x \in F^{\alpha}$. It will be shown in the proof of the next theorem that E consists of classes of Jacobi vector fields such that $J_x, \dot J_x \in F^{\alpha}$, (4).

Then, on a representative Jacobi field $J \in \mathcal{J}^{\alpha}_{\gamma,x}$, we define $\tau^{-1}(Y)$ to be the class of Jacobi fields in E/σ represented by the Jacobi field J^Y on γ which is given by $J^Y_x := \nabla_Y J$, $\dot J^Y_x := 0$. We remark that $\nabla_Y J$ is what we usually denote $\dot J$, when the parameter on γ is understood.

It is straightforward to check that $J \mapsto J^Y$ induces an isomorphism $\tau^{-1}(Y) : \sigma \to E/\sigma$ for each non-zero $J \in \sigma = \mathcal{J}^{\alpha}_{\gamma,x}/\mathcal{J}^{\gamma}_{\gamma}$. □

We remark that $V^{\alpha}_{\bar\gamma} \subset V_{\bar\gamma}$, the 4-dimensional subspace represented by Jacobi fields J such that $J, \dot J \perp \dot\gamma$. We further introduce the subspace $H^{\alpha}_{\bar\gamma,x} \subset V_{\bar\gamma}$, represented by Jacobi fields J as before, with the additional condition $J_x \in F^{\alpha}_x$. It is a 3-dimensional subspace, and it contains E_x. The curvature of $V^{\alpha}_{\bar\gamma}$ will take values in $\mathrm{Hom}(TS \otimes TS, N^V(S))$, and we show ((6) in the proof of the next theorem) that it takes values in a smaller space, $\mathrm{Hom}(TS \otimes TS, N^H(S))$. $N^V_\sigma(S) \simeq \mathrm{Hom}(\sigma, V_{\bar\gamma})/T_\sigma S$ is just the normal space of S in $\mathbb{P}(V_\gamma)$ at σ, and $N^H(S)$ is the subspace of $N^V_\sigma(S)$ consisting of elements represented by $\xi \in \mathrm{Hom}(\sigma, H^{\alpha}_{\bar\gamma,x}) \subset \mathrm{Hom}(\sigma, V_{\bar\gamma})$.

Lemma 3. *There is a canonical isomorphism*

$$\rho : N^H(S) \to \mathrm{Hom}(F^{\alpha}/T\gamma, \gamma^{\perp}/F^{\alpha}).$$

Proof. As H is a subbundle of the normal bundle $N(S)$, $N^H(S)$ is isomorphic to $\mathrm{Hom}(\sigma, H/E)$. As in Lemma 2, we will construct the inverse isomorphism ρ^{-1}. Let $\xi : F^{\alpha}/T\gamma \to \gamma^{\perp}/F^{\alpha}$ be a linear application. Let $\xi_0 : F^{\alpha} \to \gamma^{\perp}$ be a representant of ξ (it involves a choice of a complementary space to F^{α} in $\gamma^{\perp}$). We define $\rho^{-1}(\xi) \in \mathrm{Hom}(\sigma, H/E)$ as being induced by the following linear application between spaces of Jacobi fields on γ.

$\rho^{-1}(\xi) : \mathcal{J}^{\alpha}_{\gamma,x} \to \mathcal{J}^{\alpha,\perp}_{\gamma,x}$, where the second space corresponds to H_x, i.e. it contains Jacobi fields J such that $J_x \in F^{\alpha}$, $\dot J_x \perp \dot\gamma_x$. Consider a parameterization of γ around x, and let $J \in \mathcal{J}^{\alpha}_{\gamma,x}$. We define $J^{\xi} := \rho^{-1}(\xi)(J)$ by $J^{\xi}_x := 0, \dot J^{\xi}_x := \xi_0(\dot J_x)$, and it is easy to check that the class of J^{ξ} in H/E is independent of the representant ξ_0 such that ρ^{-1} is well-defined. It is also obviously invertible. □

We are now in position to translate the projective curvature of V^{α}_{γ} in terms of conformal invariants of $(\mathbf{M}, c)$.

Theorem 1. *Let x be a point on a null-geodesic γ. Then the projective curvature K of the α-cone $V^\alpha_{\tilde\gamma}$ at σ (corresponding to x, see Definition 10), which is a linear map*

$$K : T_\sigma S \otimes T_\sigma S \to N^V(S)_\sigma,$$

takes values in $N^H(S)_\sigma$ (see above), and is canonically identified with the linear map

$$K' : T_x\gamma \otimes T_x\gamma \to \mathrm{Hom}(F^\alpha_x/T_x\gamma, \gamma^\perp_x/F^\alpha_x)$$

defined by the self-dual Weyl tensor of $\mathbf{M}$*:*

$$K'(Y,Y)(X) = W^+(Y,X)Y, \quad Y \in T_x\gamma,\ X \in F^\alpha_x.$$

Proof. Consider the following analytic map, which parameterizes, locally around $x \in \gamma$, the deformations of the geodesic γ that correspond to points contained in the integral α-cone $B^\alpha_{\tilde\gamma}$:

$$f : U \to \mathbf{M},\ f(t,s,u) = \gamma^{t,s}(u),$$

where U is a neighborhood of the origin in $\mathbb{C}^3$, and $\gamma^{t,s}$ is a deformation of the null-geodesic γ, such that

$$\gamma^{t,s}(t) = \gamma(t),\ \dot\gamma^{t,s}(t) \in F^\alpha_{\gamma(t)},$$

where the parameterization of the geodesic γ satisfies $\gamma(0) = x$, and $F^\alpha_{\gamma(u)}$ is the α-plane in $T_{\gamma(u)}\mathbf{M}$ containing $\dot\gamma(u)$.

Convention. We know that f is defined around the origin in $\mathbb{C}^3$, so there exists a polydisc centered at the origin included in U, and so all the relations that we will use are true for values of the variables t, s, u sufficiently close to 0. For simplicity, we will not mention these domains.

The geodesics $\gamma^{t,s}$ correspond to points in $B^\alpha_{\tilde\gamma,\gamma(t)}$, and the Jacobi fields J^t on γ, defined as

$$J^t(u) := \partial_s f(t,0,u) \in T_{\gamma(u)}\mathbf{M},$$

correspond to vectors in $V^\alpha_{\tilde\gamma}$ tangent to the above-mentioned lines. We suppose that the deformation f is *effective*, i.e. $\partial_u\gamma^{t,s}(u) \neq 0$ and $J^t \notin \mathcal{J}^\gamma_\gamma$, which is equivalent to $\dot J^t(t) \notin T_{\gamma(t)}\gamma$. In order to compute the projective curvature of $V^\alpha_{\tilde\gamma}$, we thus need to study the (second order) infinitesimal variation of these Jacobi fields on γ. As they are determined by their value and first order derivative in $\gamma(0) = x$, we need to evaluate $\partial_t J^t(0)|_{t=o}, \partial_t \dot J^t(0)|_{t=0}$ for the first derivative of J^t at $t = 0$, and $\partial^2_t J^t(0)|_{t=0}, \partial^2_t \nabla \dot J^t(0)|_{t=0}$ for the second. Dots mean, as before, covariant differentiation with respect to the "speed" vector $\dot\gamma$; thus they correspond to the operator ∂_u.

As the covariant derivation ∇ has no torsion, we can apply the usual commutativity relations between the operators $\partial_t, \partial_s, \partial_t$ and use them to differentiate the following equation, which follows directly from the definition of f and J^t:

$$J^t(t) = 0\ \forall t. \tag{1}$$

We get then

$$\partial_t J^t(t) + \dot J^t(t) = 0, \tag{2}$$

We recall now that, besides (1), we have $\dot{J}^t(t) \in F^\alpha_{\gamma(t)}$; thus $\dot{J}^t(t)$ is isotropic, which implies that

$$\langle \partial_t \dot{J}^t(t), \dot{J}^t(t) \rangle = 0, \tag{3}$$

as $\ddot{J}^t(t) = R(\dot\gamma(t), J^t(t))\dot\gamma(t) = 0$. Equations (2) and (3) prove that

$$\partial_t J^t|_{t=0} \in \mathcal{J}^\alpha_{\gamma,x}, \tag{4}$$

which completes the proof of Lemma 2. From (3), it equally follows that $\partial_t \dot{J}^t(t)$ is isotropic, and, by differentiating (3), we get

$$\langle \partial_t^2 \dot{J}^t(t), \dot{J}^t(t) \rangle = -\langle \partial_t \ddot{J}^t(t), \dot{J}^t(t) \rangle. \tag{5}$$

From (2) we have that $\partial_t J^t(t)$ is isotropic, and also

$$\partial_t^2 J^t(t) + 2\partial_t \dot{J}^t(t) = 0,$$

which, together with (3), implies that $\partial_t^2 J^t(0)|_{t=0} \in F^\alpha_x$. Then we have

$$\partial_t^2 J^t|_{t=0} \in \mathcal{J}^{\alpha,\perp}_{\gamma,x}, \tag{6}$$

which proves that the curvature K of the α-cone takes values in $N^H(S)$, as it is represented by $\partial_t^2 J^t|_{t=0}$.

In view of Lemmas 2 and 3, it is clear now that the projective curvature K is represented by the following application:

$$(\dot\gamma, \dot\gamma, \dot{J})_x \longmapsto \partial_t^2 J^t(0)|_{t=0}.$$

From (5), as $\partial_t \ddot{J}^t(t) = R(\dot\gamma, \partial_t J^t)\dot\gamma$ and $\partial_t J^t(t) = -\dot{J}^t(t)$, we get

$$\langle K(\dot\gamma, \dot\gamma)(\dot{J}), \dot{J} \rangle = \langle R(\dot\gamma, \dot{J})\dot\gamma, \dot{J} \rangle.$$

The right-hand side actually involves only W^+, as the other components of the Riemannian curvature vanish on this combination of vectors. Thus we can replace R by W^+ in the above relation. On the other hand, the class of $W^+(\dot\gamma, \dot{J})\dot\gamma$ modulo F^α is determined by its scalar product with $\dot{J}$, which represents a non-zero generator of $F^\alpha/T\gamma$.

The proof of the theorem is now complete. □

Remark. We may ask whether the projective lines in Z are the geodesics of some projective structure. Indeed, in the conformally flat case, when $\mathbf{M}$ is the Grassmannian of 2-planes in $\mathbb{C}^4$ (the complexification of the Möbius 4-sphere), $Z \simeq \mathbb{CP}^3$, and the complex lines are given by the standard (flat) projective structure. But there are two reasons (related to each other, as we will soon see) why Z cannot carry, in general, a canonical projective structure. First, we do not necessarily have projective lines $Z_x \ni \bar\beta$ in every direction of $T_{\bar\beta}Z$ (this would mean that $\beta \simeq \mathbb{CP}^2$, see the next section for a treatment of this problem), and second, the lift of a 2-plane $F^\gamma \subset T_{\bar\beta}Z$ would be a 2-plane in $T_{\bar\gamma}B$, so $V^\alpha_{\bar\gamma}$ would be a flat cone.

Corollary 1. *The projective lines Z_x in the twistor space Z are geodesics of a projective structure iff it is projectively flat and $\mathbf{M}$ is conformally flat.*

Proof. If Z admits a projective structure, some of whose geodesics are the lines Z_x, then we have, for a fixed $\bar\beta \in Z$, a linear connection around $\bar\beta$, whose geodesics in the directions of Z_x, $\bar\beta \in Z_x$ ($\Leftrightarrow x \in \beta \subset \mathbf{M}$) coincide, locally, with Z_x. This means that the integral α-cone Z^γ, for $\gamma \subset \beta$ a null-geodesic, is part of a complex surface (namely $\exp(F^\gamma)$, where $F^\gamma \subset T_{\bar\beta}Z$ is the 2-plane corresponding to γ). Then the integral α-cone $B^\alpha_{\bar\gamma}$, the lift to B of Z^γ, is also a complex surface, and so $V^\alpha_{\bar\gamma}$ is a subset of the tangent space $T_{\bar\gamma}B^\alpha_{\bar\gamma}$, thus a flat cone. As this is true for all points of Z and for all null-geodesics γ, Theorem 1 implies that $\mathbf{M}$ is flat.

On the other hand, it is well-known that the twistor space of a conformally flat manifold admits a flat projective structure, for which the projective lines Z_x are geodesics, [1]. □

5. Compactness of null-geodesics and conformal flatness

5.1. **Complete α-cones in Z.** We have given, in the preceding section, a way to measure the projective curvature of the α-cone in B; we shall see now what happens in the special case when this cone is *complete* at a point $\bar\gamma$, i.e. when $\mathbb{P}(V^\alpha_{\bar\gamma})$ is a compact submanifold in $\mathbb{P}(T_\gamma B)$.

This situation appears for example if, for every direction in $F^\gamma \subset T_{\bar\beta}Z$, there are projective lines in Z tangent to it.

Theorem 2. *Let Z be the twistor space of the connected civilized self-dual 4-manifold $(\mathbf{M}, c)$, and suppose that, for a point $\beta \in Z$ and for a 2-plane $F^\alpha \subset T_{\bar\beta}Z$, there are projective lines Z_x tangent to each direction of F^α. Then $(\mathbf{M}, c)$ is conformally flat.*

Proof. The idea is to prove that the integral α-cone Z^γ is a smooth surface. We know that this holds at all its points except the vertex $\bar\beta$ (Proposition 4). The fact that all directions in F^γ admit a tangent line is a necessary condition for this cone to be a smooth surface, as it needs to be well-defined around $\bar\beta$.

We choose an auxiliary Hermitian (real) metric h on Z. Its restrictions h_x to the lines $Z_x \subset Z^\gamma$ yield Kählerian metrics on these lines; in fact these metrics are deformations of one another, just like the lines Z_x are. This means that the metrics h_x depend continuously on $x \in \mathbb{P}(F^\alpha)$, a parameter in a compact set. We can therefore find a lower bound $r_0 > 0$ for the injectivity radius of all (Z_x, h_x) at $\bar\beta$, and a finite upper bound R for the norm of all the second fundamental forms $H_x : TZ_x \otimes TZ_x \to (TZ_x)^\perp\,(\subset TZ)$. We can also suppose that r_0 is smaller than the injectivity radius of (Z, h) at $\bar\beta$.

The first step is to prove that Z^γ is a submanifold of class $\mathcal{C}^1$. As its tangent space is everywhere a complex subspace of TZ, it will follow that it is a complex analytic submanifold.

Consider now the exponential map $\exp_{\bar\beta} : T_{\bar\beta}Z \to Z$, defined for the metric h. If we restrict it to a ball of radius less than r_0, it is a diffeomorphism into Z. The image of the complex plane F^α is then a smooth 4-dimensional real submanifold S of Z, and there exists a positive number r_1 such that the exponential map in the directions normal to S,

$$\exp_S : TS^\perp \to Z,\, \exp(Y) := \exp_y(Y),\ y \in S,\, Y \in T_yS^\perp,$$

restricted to the vectors of length less than r_1, is a diffeomorphism.

The image of this diffeomorphism is a tubular neighborhood of S, and we will denote by $N(S, r)$ such a tubular neighborhood of "width" r, for $r < r_1$.

The existence of an upper bound R for the second fundamental forms of $Z_x, \forall x \in \gamma$, implies the following fact.

Lemma 4. *For any $r < r_1$, there is a neighborhood $U \subset T_{\bar\beta}Z$ of the origin such that $\exp(U) \cap Z^\gamma$ is contained in $N(S,r)$ and is transverse to the fibers of the orthogonal projection $p^S : N(S,r) \to S$, $p^S(\exp(Y)) := y$, where $Y \in T_yS$.*

This is standard if Z^γ is a submanifold; but it is also true in our case, where Z^γ is a union of submanifolds Z_x.

Now it is easy to prove that Z^γ is a $\mathcal{C}^1$ submanifold of Z (the projection p^S yields a local $\mathcal{C}^1$ diffeomorphism from a neighborhood of $\bar\beta$ in S to a neighborhood of $\bar\beta$ in Z^γ; it is $\mathcal{C}^1$ at $\bar\beta$ because S is tangent to Z^γ at $\bar\beta$).

So Z^γ is a $\mathcal{C}^1$ submanifold of Z. Its tangent space is complex at each point, and so Z^γ is a complex-analytic surface immersed in Z.

Then $B^\alpha_{\tilde\gamma} \subset B = \mathbb{P}(T^*Z)$, being the lift of Z^γ, is a smooth analytic surface immersed in B; in particular, the α-cone $V^\alpha_{\tilde\gamma}$ is a complex plane.

Theorem 1 implies that W^+ vanishes on the α-plane $F^\alpha_x \subset T_x\mathbf{M}$ which contains $\dot\gamma_x$, for every point $x \in \gamma$. Now, the plane $F^\gamma \subset T_{\bar\beta}Z$ is not the only one admitting projective lines Z_x tangent to any of its directions: all planes "close" to F^γ have the same property. Then W^+ vanishes on a neighborhood of γ, hence on the whole connected manifold $\mathbf{M}$. □

Remark. There is a more general situation where the integral α-cone Z^γ through β is smooth in β.

Theorem 2′. *Suppose that, for each direction $\sigma \in \mathbb{P}(T_\beta Z)$, there is a smooth (not necessarily compact) curve Z_σ, tangent to σ, such that*

(i) if σ is tangent to a projective line Z_x, then $Z_\sigma = Z_x$, and

(ii) Z_σ varies smoothly with $\sigma \in \mathbb{P}(F^\gamma)$.

Then

$$\bar Z^\gamma_\beta := \bigcup_{\sigma \in \mathbb{P}(F^\gamma)} Z_\sigma$$

is a smooth surface around β containing the α-cone Z^γ, and $W^+(F^\gamma_x) = 0$, $\forall x \in \gamma$, where $F^\gamma_x \subset T_x\mathbf{M}$ is the α-plane containing $\dot\gamma$.

The proof is similar to that of the previous theorem. Note that, if there is a direction σ which is not tangent to a projective line Z_x, we cannot apply the deformation argument in Theorem 2 to conclude that W^+ vanishes everywhere.

Example. If $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \smallsetminus \mathcal{F}$, then $Z = \mathcal{F}$ and there are some particular planes for which the conditions in Theorem 2′ are satisfied, although Theorem 2 never applies to Z: for a generic 2-plane F^γ, the α-cone V^α_γ is not flat. These particular planes in TZ correspond to the vanishing of W^+ on some particular α-planes, but $\mathbf{M}$ is not anti-self-dual (see Section 7.3, and also 7.7, 7.8).

The following result is a direct consequence of Theorem 2:

Theorem 3. *If a civilized self-dual complex 4-manifold $(\mathbf{M}, c)$ admits a compact null-geodesic, then the conformal structure of $\mathbf{M}$ is flat, and the null-geodesic is simply-connected.*

We simply have to use the fact that a null-geodesic γ of a civized self-dual manifold identifies with an open set of $\mathbb{P}(F^\gamma)$, where F^γ is the associated 2-plane in $T_{\bar\beta}Z$, where $\beta \supset \gamma$.

The condition that $\mathbf{M}$ be civilized is not essential, if we assume that γ is simply-connected (and compact); in order to prove that, we need to cover γ with civilized (e.g. geodesically connected) open sets U_i, and relate the local twistor spaces $Z_i := Z(U_i)$; the key point is that, if γ is diffeomeorphic to $\mathbb{CP}^1$, it turns out that a neighbourhood of $\bar{\beta} \in Z_i$—for β the β-surface containing γ—can be identified with the space of *deformations of* γ *as a compact curve* ([2], Proposition 5). Then we conclude, using the criterion from Theorem 2′ and a deformation argument, that $\mathbf{M}$ is conformally flat.

This method is used in [2] to prove the same thing starting from a conformal complex 3-manifold (using the *LeBrun correspondence*, i.e. the local realization of a conformal 3-manifold as the conformal infinity of a (germ-unique) self-dual manifold [12]), but we also show there, by different methods, that, in all generality, a conformal complex n-manifold ($n \geq 4$) containing a compact, simply-connected null-geodesic is conformally flat ([2], Theorem 4).

6. The projective structure of β-surfaces in a self-dual manifold

The null-geodesics contained in a β-surface β define a *projective structure* on the totally-geodesic surface β, which is also given by any connection on β induced by a Levi-Civita connection on $\mathbf{M}$. We claim that this projective structure is *flat*, i.e. locally equivalent to $\mathbb{CP}^2$.

Example. If $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \smallsetminus \mathcal{F}$, then a β-surface indexed by $(L, l) \in \mathcal{F}$ is $\beta^{(L,l)} = \{(A, a) | A \subset l, L \subset a, A \not\subset a\} \simeq \mathbb{C}^2$, and the null-geodesics in $\beta^{(L,l)}$ are identified with the affine lines in $\mathbb{C}^2$ (see Section 7.5).

To prove the projective flatness of a 2-dimensional manifold β, we need to prove that the *Thomas tensor* T vanishes identically [20]. This tensor is an analog of the *Cotton-York tensor* in conformal geometry (there is also a *Weyl tensor* of a projective structure, but it only appears in dimensions greater than 2).

For a connection ∇ in the projective class of β, the Thomas tensor is defined as follows [20]: For $X, Y, Z \in T\beta$,

$$\begin{aligned} T(X,Y,Z) := &-2(\nabla_Z K)(Y)X + 2(\nabla_Y K)(Z)X \\ &- (\nabla_Z K)(X)Y + (\nabla_Y K)(X)Z, \end{aligned} \tag{7}$$

where the derivation involves only the curvature term K, which is defined by $K(Y)X := \operatorname{tr} R(Y, \cdot)X$, the trace of the endomorphism $R(Y, \cdot)X \in \operatorname{End}(T\beta)$.

The Thomas tensor is independent of the connection ∇. Therefore we will consider that ∇ is induced by a Levi-Civita connection on $\mathbf{M}$.

Proposition 6. *The Thomas tensor of a β-surface can be expressed in terms of the anti-self-dual Cotton-York tensor of* $\mathbf{M}$. *Thus it is identically zero.*

Proof. First we need to define the *anti-self-dual* Cotton-York tensor as an irreducible component of the Cotton-York tensor of $\mathbf{M}$.

Convention. We denote by C the Cotton-York tensor of $(\mathbf{M}, c)$; we will not use this letter for the isotropic cone in this section.

The Cotton-York tensor is not conformally invariant; its definition depends on a (local) metric g in the conformal structure, which is supposed to be fixed [5]:

$$C(X,Y)(Z) := (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \ \forall X,Y,Z \in T\mathbf{M}, \tag{8}$$

where h is the normalized Ricci tensor of $\mathbf{M}$,

$$h = \frac{1}{2n(n-1)} \operatorname{Scal} \cdot g + \frac{1}{n-2} \operatorname{Ric}_0, \tag{9}$$

$\mathrm{Ric}_0, \mathrm{Scal}$ being the trace-free Ricci tensor, resp. the scalar curvature of the metric g, and $n := \dim\mathbf{M}$. In our case $n = 4$, but the formula applies in all dimensions greater than 2 [5].

Remark. The Cotton-York tensor C of $\mathbf{M}$ is a 2-form with values in $T^*\mathbf{M}$; thus it has two components, $C^+ \in T^*\mathbf{M} \otimes \Lambda^+\mathbf{M}$, and $C^- \in T^*\mathbf{M} \otimes \Lambda^-\mathbf{M}$. C satisfies a *first* Bianchi identity, due to the fact that h is a symmetric tensor, and also a *contracted* (second) Bianchi identity, which comes from the second Bianchi identity in Riemannian geometry, [5]:

$$\sum C(X,Y)(Z) = 0, \quad \text{circular sum;} \tag{10}$$

$$\sum C(X,e_i)(e_i) = 0, \quad \text{trace over an orthonormal basis.} \tag{11}$$

That means that $C \in \Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$, and is orthogonal on $\Lambda^3\mathbf{M} \subset \Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$ and on $\Lambda^1\mathbf{M}$, which is identified with the image in $\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$ by the metric adjoint of the contraction (11).

Now, the Hodge operator $* : \Lambda^2\mathbf{M} \to \Lambda^2\mathbf{M}$ induces a symmetric endomorphism of $\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M}$, which maps the two above spaces isomorphically into each other. This implies that C^+ and C^- satisfy (10) and (11) (note that these two relations are equivalent in their case) and are, therefore, $SO(4,\mathbb{C})$-irreducible.

The Cotton-York tensor is related to the Weyl tensor of $\mathbf{M}$ by the formula [5]

$$\delta W = C, \tag{12}$$

where $\delta : \Gamma(\Lambda^2\mathbf{M} \otimes \Lambda^2\mathbf{M}) \to \Gamma(\Lambda^2\mathbf{M} \otimes \Lambda^1\mathbf{M})$ is induced by the codifferential on the second factor, and by the Levi-Civita connection ∇ on the first. Then, C^+ has to be the component of δW in $\Lambda^+\mathbf{M} \otimes \Lambda^1\mathbf{M}$, and we know that the restriction of W^- to $\Lambda^+\mathbf{M} \otimes \Lambda^2\mathbf{M}$ is identically zero. This means that

$$\delta W^+ = C^+, \text{ and also} \tag{13}$$

$$\delta W^- = C^-. \tag{14}$$

Hence, as $\mathbf{M}$ is self-dual, C^- vanishes identically.

We can prove now that the Thomas tensor of a β-surface β is identically zero. First we show that

$$K(Y)X = \mathrm{tr}|_{T\beta} R(Y,\cdot)X = h(X,Y),\ \forall X,Y \in T\beta. \tag{15}$$

We recall from [5] that the *suspension* $h \wedge \mathbf{I}$ of h by the identity, viewed as an endomorphism of $\Lambda^2\mathbf{M}$, is defined by

$$(h \wedge \mathbf{I})(X,Y) := h(X) \wedge Y - h(Y) \wedge X,\ \ X,Y \in T\mathbf{M}, \tag{16}$$

where h is identified with a symmetric endomorphism of $T\mathbf{M}$.

We have then the following decomposition of the Riemannian curvature [5]:

$$R = h \wedge \mathbf{I} + W^+ + W^-.$$

Of course, if $\mathbf{M}$ is self-dual, then $W^- = 0$ and $W^+(X,Y) = 0$ if $X,Y \in T\beta$ (in fact, the elements in $\Lambda^2 F^\beta$, for any β-plane $F^\beta \subset T_x\mathbf{M}$, correspond to the isotropic vectors in $\Lambda^-\mathbf{M}$), because $W^+|_{\Lambda^- M} = 0$. Then, if we choose the basis $\{X,Y\}$ in

$T\beta$, we get

$$\begin{aligned} K(Y)X &= \operatorname{tr}|_{T\beta}(h \wedge \mathbf{I})(Y,\cdot)X \\ &= \text{the component along } X \text{ of } (h \wedge \mathbf{I})(Y,X)X \\ &= h(Y,X), \end{aligned}$$

which proves (15). The Thomas tensor of the projective structure of β has the following expression (see (7)):

$$T(X,Y,Z) = -3(\nabla_Z h)(Y,X) + 3(\nabla_Y h)(Z,X) = 3C(Y,Z)(X),\ \forall X,Y,Z \in T\beta,$$

and, as $C^+(\cdot,\cdot)(X)$ vanishes on the anti-self-dual 2-form $Y \wedge Z$, we conclude that

$$T(X,Y,Z) = C^-(Y,Z)(X) = 0. \tag{17}$$

□

As the flatness of the projective structure on a 2-dimensional manifold is equivalent to the vanishing of its Thomas tensor [20], we get

Corollary 2. *The projective structure of the β-surfaces of a self-dual complex manifold* $\mathbf{M}$ *is flat.*

From the classification of projectively flat compact complex surfaces ([9], see also [7]), we then get a classification of compact β-surfaces in $\mathbf{M}$:

Theorem 4. *A compact β-surface of a self-dual complex 4-manifold belongs (up to finite covering) to one of the following classes:*

(1) $\mathbb{CP}^2$*;*
(2) *a compact quotient of the complex-hyperbolic plane* $\mathbf{H}^2_{\mathbb{C}}/\Gamma$*;*
(3) *a compact complex surface admitting a (flat) affine structure:*
  (i) *a Kodaira surface;*
  (ii) *a properly elliptic surface with* b_1 *odd;*
  (iii) *an affine Hopf surface;*
  (iv) *an Inoue surface;*
  (v) *a complex torus.*

See [7], [9], [11] for details.

7. Examples

7.1. **The flat case.** The first example is the "flat" case: $Z = \mathbb{CP}^3 = \mathbb{P}(\mathbb{C}^4)$, with its canonical projective structure and its space of projective lines $\mathbf{M} = \mathrm{Gr}(2,\mathbb{C}^4)$. ($Z$ is equally the twistor space of the Riemannian round 4-sphere, which is, therefore, a real part of $\mathrm{Gr}(2,\mathbb{C}^4)$.) If $\beta \in Z$, then the β-surface associated to it is the set $\{x \in \mathrm{Gr}(2,\mathbb{C}^4) | \beta \subset x \subset \mathbb{C}^4\}$. In this flat case, we can equally define the α-twistor space Z^*, which is the dual projective 3-space $(\mathbb{CP}^3)^* := \mathbb{P}((\mathbb{C}^4)^*) = \mathrm{Gr}(3,\mathbb{C}^4)$, and an α-surface $\alpha \in Z^*$ is the set $\{x \in \mathrm{Gr}(2,\mathbb{C}^4) | x \subset \alpha \subset \mathbb{C}^4\} \subset \mathbf{M}$. A null-geodesic γ is then determined by a pair of *incident* isotropic surfaces α and β such that $\alpha \cap \beta = \gamma$, where α is an α-surface and β is a β-surface; *incident* means (see above) that β, seen as a line in $\mathbb{C}^4$, is *included* in α, seen as a 3-plane in $\mathbb{C}^4$. γ is then the following set of points in $\mathbf{M}$:

$$\gamma = \{x \in \mathrm{Gr}(2,\mathbb{C}^4) | \beta \subset x \subset \alpha\}.$$

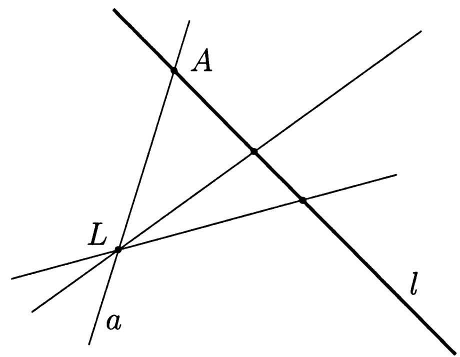

FIGURE 4.

α-surfaces and β-surfaces are diffeomorphic to $\mathbb{CP}^2$, null-geodesics to $\mathbb{CP}^1$, and the ambitwistor space B is the "partial flag" manifold

$$B = \{(\alpha, \beta) \in (\mathbb{CP}^3)^* \times \mathbb{CP}^3 | \beta \subset \alpha\}.$$

The flag manifold, of dimension 7, is isomorphic to the total space $\mathbb{P}(C)$ of the projective cone bundle over $\mathbf{M}$.

7.2. $\mathbb{CP}^2$**.** Another example is when Z is the twistor space of the real Riemannian manifold $\mathbb{CP}^2$, with the Fubini-Study metric. Then Z is the manifold of flags in $E = \mathbb{C}^3$, $\mathcal{F} := \{(L, l) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | L \subset l\}$ ($\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$ are viewed as the space of lines, resp. 2-planes, in E) [1]. A projective line Z_x in Z is a set

$$Z_x = \{(L, l) \in \mathcal{F} | L \subset a^x,\ A^x \subset l\}$$

(see Figure 4), where (A^x, a^x) belongs to $\mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$, which is, therefore, the space $\mathbf{M}$ of such lines, and a conformal self-dual 4-manifold. It can be naturally compactified within the space of analytic cycles of Z to $\overline{\mathbf{M}} = \mathbb{P}(E) \times \mathbb{P}(E)^*$, which is obviously a smooth manifold, but it carries no global conformal structure, as its canonical bundle has no square root. This means that the conformal structure on $\overline{\mathbf{M}}$ is smooth on $\mathbf{M}$, and *singular* on $\mathcal{F} = \overline{\mathbf{M}} \setminus \mathbf{M}$. The cycles of Z corresponding to a point $\bar{x} = (A, a)$ in this subset are pairs of complex projective lines in Z:

$$Z_{\bar{x}} = \{(A, l) \in Z = \mathcal{F}\} \cup \{(L, a) \in Z = \mathcal{F}\}.$$

A β-surface in $\mathbf{M}$, corresponding to a point $\beta = (L, l) \in Z$, is the set

$$\beta = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l,\ L \subset a,\ A \neq L,\ a \neq l\},$$

and can be naturally compactified to

$$\bar{\beta} = \{(A, l^\beta) \in \mathcal{F}\} \times \{(L^\beta, a) \in \mathcal{F}\} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1.$$

7.3. **The tangent space to $\mathcal{F}$.** In order to describe the null-geodesics of $\mathbf{M}$ as 2-planes in Z, we first study the tangent space of $Z = \mathcal{F}$ at $\beta = (L, l)$.

A vector in $T_{(L,l)}\mathcal{F}$ is a pair of vectors (V, v), with $V \in T_L\mathbb{P}(E)$ and $v \in T_l\mathbb{P}(E)^*$, which satisfy a linear condition (as $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$). Actually, there is a duality

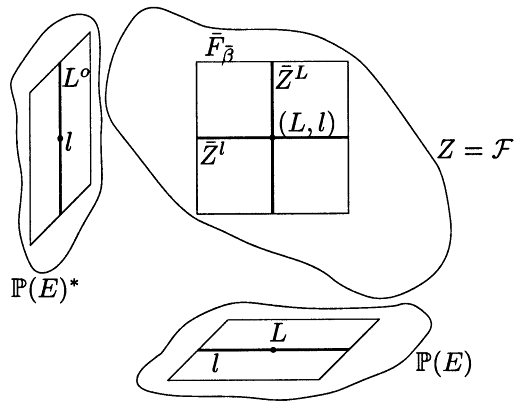

FIGURE 5.

between $\mathbb{P}(E)^*$, the Grassmannian of 2-planes in E, and $\mathbb{P}(E^*)$, the projective space of $E^* := \mathrm{Hom}(E, \mathbb{C})$, and an analogous one between $\mathbb{P}(E)$ and $\mathbb{P}(E^*)^*$:

$$\mathbb{P}(E)^* \ni l \overset{\simeq}{\longmapsto} l^o \in \mathbb{P}(E^*),$$
$$\mathbb{P}(E) \ni L \overset{\simeq}{\longmapsto} L^o \in \mathbb{P}(E^*)^*.$$

Then the flag manifold $\mathcal{F}$ is defined, as a submanifold of $\mathbb{P}(E) \times \mathbb{P}(E)^*$, by the equation

$$y(Y) = 0, \ \forall y \in l^o, \forall Y \in L.$$

The vector $V \in T_L\mathbb{P}(E)$ is an element in $\mathrm{Hom}(L, E/L)$. By duality, $v \simeq v^o \in \mathrm{Hom}(l^o, E^*/l^o)$. Then the vector $(V, v) \in T_{(L,l)}\mathbb{P}(E) \times \mathbb{P}(E)^*$ lies in $\mathcal{F}$ iff

$$v^o(y^o)(Y) + y^o(V(Y)) = 0, \ \forall Y \in L, \forall y^o \in l^o, \tag{18}$$

or, equivalently,

$$v|_L = \pi_l \circ V, \tag{19}$$

where $\pi_l : E/L \to E/l$ is the projection (as $L \subset l$).

The geometry of $\mathcal{F}$, as a subset of $\mathbb{P}(E) \times \mathbb{P}(E)^*$, can be described as in Figure 5.

7.4. **The 2-planes in $\mathcal{F}$.** Let us consider now a 2-plane F in $T_{(L,l)}\mathcal{F}$, and the cycles (corresponding to points in $\overline{\mathbf{M}}$) tangent to it. We have three cases:

1. $F = \bar{F}_\beta$ is the "degenerate" 2-plane tangent to the 2 special curves $\bar{Z}_L$, $\bar{Z}_l$ whose union is the special cycle $\bar{F}(L, l)$ corresponding to $(L, l) \in \overline{\mathbf{M}} \setminus \mathbf{M}$. There are no projective lines Z_x, $x \in \mathbf{M}$, tangent to it; only the special cycles $\bar{Z}_{(L,a)}$, $L \subset a$, and $\bar{Z}_{(A,l)}$, $A \subset l$, are tangent to $\bar{F}_{(L,l)}$, actually only to the two privileged directions of $\bar{Z}_L$, resp. $\bar{Z}_l$.

Remark. The special curves $\bar{Z}_L$, $\bar{Z}_l$ have trivial normal bundle, being fibers of the projections from $\mathcal{F}$ to $\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$, so these special curves form two complete families of analytic cycles in $\mathcal{F}$, isomorphic to $\mathbb{P}(E)$, resp. $\mathbb{P}(E)^*$. Two such curves are incident iff they are of different types ($\bar{Z}_L$ is *of type* E, $\bar{Z}_l$ is *of type* E^*), so they can only form "polygons" with an even number of edges. But there are no quadrilaterals, as one can easily check, using the fact that $\bar{Z}_L$ and $\bar{Z}_l$ are incident

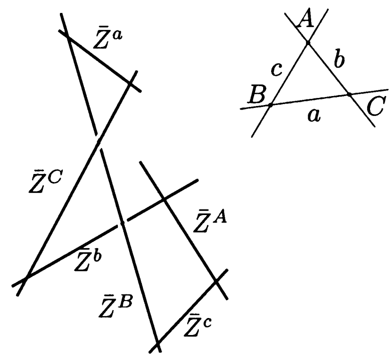

FIGURE 6.

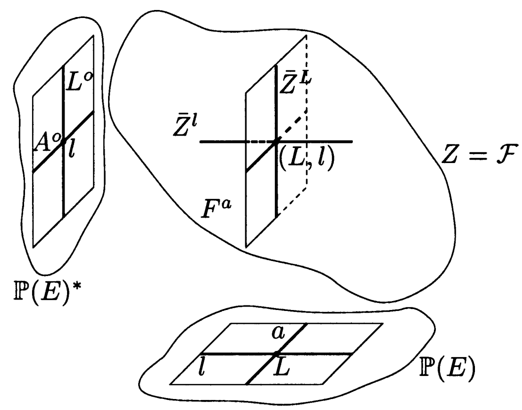

FIGURE 7.

iff $L \subset l$, thus iff l is a line in $\mathbb{P}(E)$ containing L. On the other hand, there are hexagons, corresponding to the 3 vertices and 3 sides of a triangle in $\mathbb{P}(E) \simeq \mathbb{CP}^2$ (see Figure 6).

The above hexagon is not "flat", i.e. there is no canonical submanifold of $\mathcal{F}$ containing it. This, and the fact that there are no quadrilaterals made of $\bar{Z}$-type curves, is just a consequence of the fact that the distribution $\bar{F}$ on $Z = \mathcal{F}$ is *not integrable*; in fact it is the holomorphic *contact structure* induced by the Fubini-Study *Einstein metric* on $\mathbb{CP}^2$ ([3]; see also Section 7.6).

2. $F = F^a$, for $a \supset L$, $a \neq l$. This is a 2-plane that is tangent to only one of the special curves $\bar{Z}_L$. The projective lines tangent to F^a at $\beta = (L, l)$ are $Z_{(A,a)}$, $\forall A \subset l$, $A \neq L$; hence the corresponding null-geodesic is

$$\gamma^a = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l,\ A \neq l\}, \tag{20}$$

thus it is diffeomorphic to $\mathbb{C}$, and its closure is

$$\bar{\gamma}^a = \{(A, a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l\} \simeq \mathbb{CP}^1$$

(see Figure 7).

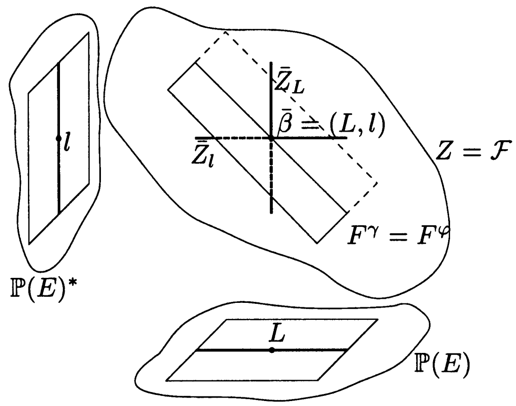

FIGURE 8.

Remark. The "limit" curve is $\bar{Z}_{(L,a)}$, so it is non-singular at (L,l). Actually, the points of $Z_{(A,a)}$ *close* to (L,l) converge, when $A \to L$, to some points in $\bar{Z}_L$, which is tangent to F^a. We can then apply the same method as in Theorem 2 to conclude that the integral α-cone associated to F^a is a smooth manifold *around* (L,l); thus, from Theorem 1, the Weyl tensor W^+ of $\mathbf{M}$ vanishes on the β-planes generated along γ^a by its own direction. We will see that the vanishing of W^+ on these α-planes leads to the existence of some α-surfaces, see below. Of course, the deformation argument in Theorem 2 does not hold in the present case, as the normal bundle of $\bar{Z}_L$ is trivial, thus different from that of the rest of the rational curves $Z_{(A,a)}$ (as we will see below, generic 2-planes through (L,l) do not admit projective lines tangent to all their directions).

2′. We have a similar situation for planes $F = F^A$ — $A \subset l$, $A \neq L$, tangent to the other special curve $\bar{Z}_l$.

3. This is the generic case: $F = F^\varphi$, where $\varphi : \mathbb{P}(l) \to \mathbb{P}(L^o)$ is a projective diffeomorphism such that $\varphi(L) = l^o$. Indeed, the tangent spaces $T_L\mathbb{P}(E)$ and $T_l\mathbb{P}(E)^*$ are isomorphic to $\mathrm{Hom}(L^o, E^*/L^o)$, resp. to $\mathrm{Hom}(l, E/l)$, and a generic 2-plane F in $T_{(L,l)}\mathcal{F}$ is the graph of a linear isomorphism $\phi : T_L\mathbb{P}(E) \to T_l\mathbb{P}(E)^*$ satisfying a linear condition (18) or (19). Actually, the graph is determined by the projective application φ induced by ϕ from $\mathbb{P}(T_L\mathbb{P}(E)) \simeq \mathbb{P}(L^o)$ to $\mathbb{P}(T_l\mathbb{P}(E)^*) \simeq \mathbb{P}(l)$ (see Figure 8).

The condition $\varphi(L) = l^o$ is implied by (19). The null-geodesic associated to the 2-plane F^φ is

$$\gamma^\varphi = \{(A,a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* \smallsetminus \mathcal{F} | A \subset l,\, a^o \subset L^o\, a^o = \varphi(A)\}, \tag{21}$$

and its closure in $\overline{\mathbf{M}}$ is

$$\bar{\gamma}^\varphi = \{(A,a) \in \mathbb{P}(E) \times \mathbb{P}(E)^* | A \subset l,\, a^o \subset L^o\}. \tag{22}$$

Hence the "limit" point is $(L,l) \in \overline{\mathbf{M}}$, corresponding to the special cycle $\bar{Z}_{(L,l)}$, none of whose components is tangent to F^φ. The integral α-cone associated to F^φ looks like what is shown in Figure 9.

7.5. **The null-geodesics of the complexification of $\mathbb{CP}^2$.** The application φ has the following interpretation in terms of projective geometry on $\mathbb{CP}^2 = \mathbb{P}(E)$: a

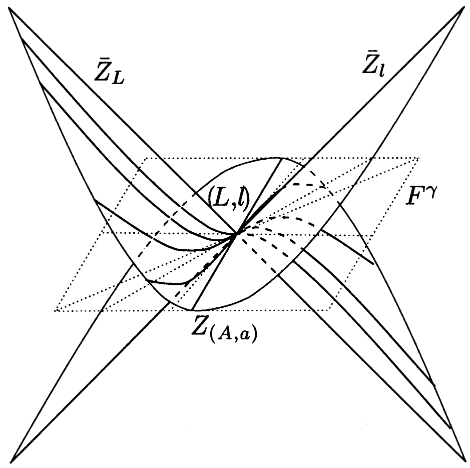

FIGURE 9.

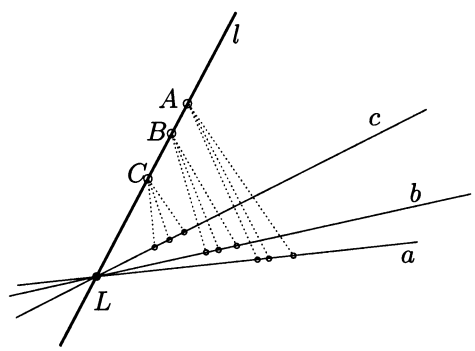

FIGURE 10.

direction $\mathbb{C}v$ in $T_l\mathbb{P}(E)^*$ is identified with the point $\ker v \equiv A \in l/L \subset \mathbb{P}(E)$ and a direction $\mathbb{C}V \subset T_L\mathbb{P}(E)$ is identified with a direction (thus a projective line a) through $L \in \mathbb{P}(E)$. φ is, thus, a *homography* that associates to $A \in l$ (we identify l with the projective line $l/L \subset \mathbb{P}(E)$) the line $a \ni L$. As $\varphi(L) = l$, we have, then, that three points $(A,a),(B,b),(C,c) \in \beta^{(L,l)}$ belong to the same null-geodesic iff

$$(A, B : C, L) = (a, b : c, l), \tag{23}$$

i.e. the cross-ratio of the points $A, B, C, L \in l$ equals the cross-ratio of the lines a, b, c, l through L (the dotted lines, together with their intersections with the lines a, b, c, correspond to the points in the integral α-cone; see Figure 10).

We can now describe the null-geodesics passing through a point $(A,a) \in \mathbf{M}$ and contained in a β-surface $\beta^(L,l)$ whose closure $\bar{\beta}$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$: they coincide with the rational curves in $\bar{\beta}$ containing (A,a); except the "horizontal" $(\bar{\gamma}^A)$ and "vertical" $(\bar{\gamma}^a)$ ones, all these curves contain (L,l); see Figure 11.

We remark that, in the usual affine coordinates on

$$\beta \simeq (\mathbb{CP}^1 \setminus \{L\}) \times (\mathbb{CP}^1 \setminus \{l\}) \simeq \mathbb{C}^2,$$

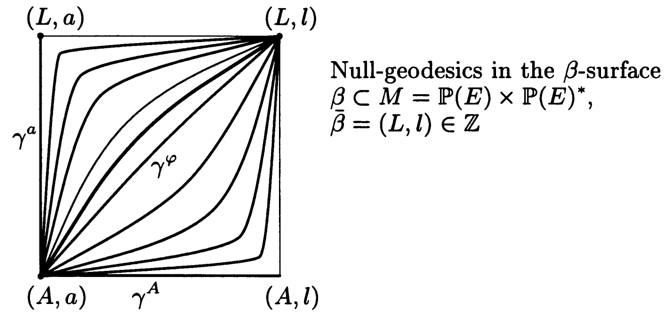

FIGURE 11.

these null-geodesics are the affine lines containing (A, a); thus the *projective structure* on β is (locally) isomorphic to a flat affine structure. We have seen, in Section 6 (Corollary 2), that this is true for all β-surfaces of a self-dual manifold.

7.6. **The conformal structure of the complexification of $\mathbb{CP}^2$.** Now let us study the conformal structure of $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ directly; actually $\mathbf{M}$ has a complex metric g. Let $(A, a) \in \mathbf{M}$; then A is transverse to a, and so we have the isomorphisms $E/a \simeq A$ and $E/A \simeq a$. Then, a vector $(V, v) \in T_{(A,a)}\mathbf{M}$ is identified with a pair of homomorphisms $V : A \to a$ and $v : a \to A$, and the metric g is given by

$$g((V, v), (W, w)) := \operatorname{tr}(v \circ W + w \circ V), \quad \forall (V, v), (W, w) \in T_{(A,a)}\mathbf{M}. \tag{24}$$

Remark (The real part). Let h be an Hermitian metric on E. Then we have a real-analytic embedding of $\mathbf{M}_0 \simeq \mathbb{P}(E)$ into $\mathbf{M}$, given by

$$\mathbb{P}(E) \ni A \longmapsto (A, A^\perp) \in \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}.$$

A vector $(V, v) \in T_{(A,A^\perp)}\mathbf{M}$ is tangent to $\mathbf{M}_0$ iff

$$h(x, v(y)) + h(V(x), y) = 0, \ \forall x \in A, \ \forall y \in A^\perp.$$

Then one easily checks that

$$g((V, v), (W, w)) = -2h(V, W), \ \forall (V, v), (W, w) \in T_{(A,A^\perp)}\mathbf{M}_0.$$

Hence, up to a constant, the restriction of g to $\mathbf{M}_0 \simeq \mathbb{CP}^2$ is the Fubini-Study metric of $\mathbb{CP}^2 \simeq S^5/S^1$.

An isotropic vector in $\mathbf{M}$ is $(V, v) \in T_{(A,a)}\mathbf{M}$, with $v \circ V = 0$, viewed as an endomorphism of A (see above), or, equivalently, with

$$\dim(A + V(A) \cap \ker v) > 0. \tag{25}$$

Let us see which is the limit of the isotropic cone in the points of $\mathcal{F}$: from the relation above, it follows that the isotropic cone at a point $x \in \mathcal{F}$ is

$$C_x = \{(0, v) \in T_x\mathcal{F}\} \cup \{(V, 0) \in T_x\mathcal{F}\},$$

so the conformal structure of $\mathbf{M}$ is *singular* at the "infinity" $\mathcal{F}$.

Remark. The situation $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$ is very similar to the one treated in [2]; see also [12]: $\mathbb{P}(E) \times \mathbb{P}(E)^*$ has an Einstein self-dual metric g, singular at the "infinity", and this Einstein structure yields a contact structure on the twistor space $Z = \mathcal{F}$; the field of 2-planes determined by this contact structure corresponds to the "infinity" $\mathcal{F} \subset \mathbb{P}(E) \times \mathbb{P}(E)^*$. But these planes do not admit tangent rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$: the conformal structure does not extend to the "infinity" (which is, therefore, not a *conformal infinity*).

7.7. **α-planes and β-planes.** We consider the isotropic planes in $T_{(A,a)}\mathbf{M}$ ($A \not\subset a$): For a fixed isotropic direction, represented by a generic vector $(V, v) \in T_{(A,a)}\mathbf{M}$, the line $\ker v \subset a$ and the plane $V(A) + A \supset A$ are fixed. The linear space of all vectors $(W, v) \in T_{(A,a)}\mathbf{M}$ satisfying

$$W(A) \subset A + V(A), \quad w|_{\ker v} = 0,$$

is isotropic and orthogonal to (V, v): they form a β-plane. The α-plane F^α containing (V, v) corresponds to the isotropic vectors (W, w) orthogonal to (V, v) with $\ker w \neq \ker v$. As a plane transverse to all the β-planes (whose projection onto $T_A\mathbb{P}(E)$ or $T_a\mathbb{P}(E)^*$ is never injective), F^α is determined by a linear isomorphism $\varphi : T_A\mathbb{P}(E) \to T_a\mathbb{P}(E)^*$, whose graph in $T_{(A,a)}\mathbb{P}(E) \times \mathbb{P}(E)^*$ is F^α; φ induces the application $\mathbb{P}\varphi : \mathbb{P}(a) \to \mathbb{P}(E/A)$ between the projective spaces of $T_A\mathbb{P}(E)$, resp. $T_a\mathbb{P}(E)^*$. The plane $F^\alpha = F^\varphi$, the graph of φ, is isotropic iff $V \subset \mathbb{P}\varphi(V)$, $\forall V \in \mathbb{P}(a)$, i.e. $\mathbb{P}\varphi$ is the homography that sends a point X in a into the projective line through A and X. We can extend φ to a projective isomorphism $\varphi' : \mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E)) \to \mathbb{P}(\mathbb{C} \oplus T_a\mathbb{P}(E)^*)$: for example, $\mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E))$ contains $T_A\mathbb{P}(E)$ as an affine open set. Then φ' is defined as follows:

$$\begin{aligned} \varphi'|_{T_A\mathbb{P}(E)} &:= \varphi, \\ \varphi'|_{\mathbb{P}(T_A\mathbb{P}(E))} &:= \mathbb{P}\varphi. \end{aligned}$$

Actually $\mathbb{P}(\mathbb{C} \oplus T_A\mathbb{P}(E)) \simeq \mathbb{P}(E)$ and $\mathbb{P}(\mathbb{C} \oplus T_a\mathbb{P}(E)^*) \simeq \mathbb{P}(E)^*$. We then have

Proposition 7. *A generic α-plane $F^\alpha = F^\varphi$ in $T_{(A,a)}\mathbf{M}$ is the graph of a linear isomorphism $\varphi : T_A\mathbb{P}(E) \to T_a\mathbb{P}(E)^*$, which is determined by a projective isomorphism*

$$\varphi' : \mathbb{P}(E) \to \mathbb{P}(E)^*$$

such that $\varphi'(A) = a$ and $\varphi'(l) = l \cap a$, for all $l \supset A$.

7.8. **Exponentials of α-planes.** The exponential $\exp(F^\varphi)$ has an interpretation in terms of projective geometry. Each direction $\mathbb{C}(V, v) \subset F^\varphi$ is determined by the point $\ker v$ in $a \subset \mathbb{P}(E)$ and the line through A and $\ker v$, and a homography $\phi^{(V,v)}$ from the points B of the projective line $A + \ker v$ to the space of lines b through $\ker v$ (see Figure 12 and the convention below). As this homography is the restriction of φ' to the appropriate spaces, it follows that it is related to the homography $\phi^{(W,w)}$, where $\mathbb{C}(W, w)$ is another direction in F^φ: the points $D := b \cap c$, $P := a \cap (B + C)$ and A are collinear (see Figure 12).

Of course, this implies that P determines a homography ψ^P between the lines $A + \ker v$ and $A + \ker w$, such that $\psi^P(A) = A$ and $\psi^P(\ker v) = \ker w$. Then, for any other points $B' \in (A + \ker v)$, $C' = \psi^P(B) \in (A + \ker w)$, the lines $b' = \phi^{(V,v)}(B')$, $c' = \phi^{(W,w)}(C')$ intersect on the line $(A + P)$ (see the right-hand side of Figure 12).

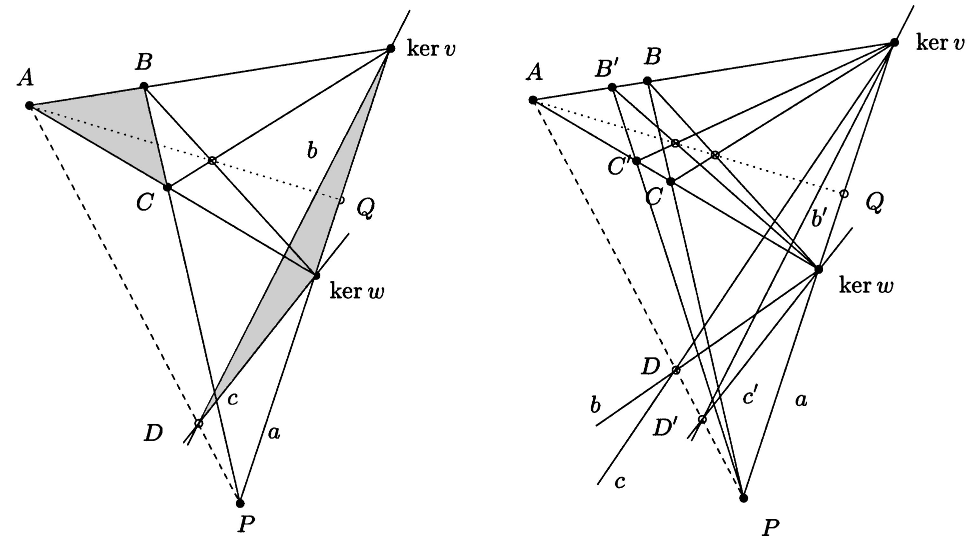

FIGURE 12.

Convention. In the framework of plane projective geometry, we identify a point in $\mathbb{P}(E)^*$ with a line in $\mathbb{P}(E)$ (we denote, for example, $\ker v \in a$). The lines determined by the distinct points B and C will be denoted by $(B+C)$ (thus $B, C \in (B+C)$).

The null-geodesic tangent to (V, v) at (A, a) is the set $\{(B, b) | B \in (A+\ker v),\ b = \phi^{(V,v)}(B)\}$, and the null-geodesic tangent to (W, w) is the analogous set of the pairs (C, c). Thus

$$\exp_{(A,a)}(F^\varphi) = \exp_{(A,a)}(F^\alpha) = \{(C, c) | C \in \mathbb{P}(E),\ C \neq A,\\ c = ((C+A) \cap a) + ((A+P) \cap b^C)\} \cup \{(A, a)\},$$

where $B^C := (A + \ker v) \cap (P + C)$, and $b^C := \phi^{(V,v)}(B^C)$, as in Figure 12 (where $B = B^C$, $b = b^C$). This gives the exponential of the α-plane determined by the isotropic vector (V, v). We remark that the point (A, a) has a privileged position in $\exp_{(A,a)}(F^\alpha)$: $(a \cap b) \in (A + B)\ \forall (B, b) \in \exp_{(A,a)}(F^\alpha)$; on the other hand, $(b \cap c) \notin (B + C)$ in general (see Figure 12), which means that the points (B, b) and (C, c) are not *null-separated* (i.e. they do not belong to the same null-geodesic). That means that $\exp_{(A,a)}(F^\alpha)$ is not totally isotropic; thus there is no α-surface tangent to a generic α-plane—not surprising, as the corresponding α-cone is not flat (see Section 7.4).

But there are α-surfaces tangent to the two α-planes $\{(V, 0) | V \in T_A\mathbb{P}(E)\}$ and $\{(0, v) | v \in T_a\mathbb{P}(E)^*\}$: the “slices” $\{A\} \times \mathbb{P}(E)^*$ and $\mathbb{P}(E) \times \{a\}$. (It is easy to see that these planes are isotropic, and that they are not β-planes, as these project on *lines* in $T_A\mathbb{P}(E)$, resp. $T_a\mathbb{P}(E)^*$.)

Thus $\mathbf{M} = \mathbb{P}(E) \times \mathbb{P}(E)^* \setminus \mathcal{F}$ is a conformal self-dual manifold, not anti-self-dual, that admits α-surfaces passing through any point.

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Centre de Mathématiques, UMR 7640 CNRS, Ecole Polytechnique, 91128 Palaiseau cedex, France

E-mail address: belgun@math.polytechnique.fr

Current address: Mathematisches Institut, Augustusplatz 10/11, 04109 Leipzig, Germany

E-mail address: Florin.Belgun@math.uni-leipzig.de